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Differentiability properties of the minimal average action

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Abstract. Given a $\mathbf{Z}^{n+1}$-periodic variational principle on $\mathbf{R}^{n+1}$ we look for solutions $u : \mathbf{R}^n \to \mathbf{R}$ minimizing the variational integral with respect to compactly supported variations. To every vector $\alpha \in \mathbf{R}^n$ we consider a subset $\mathscr{M}_\alpha$ of solutions which have an average slope α when averaging over $\mathbf{R}^n$. The minimal average action $A(\alpha)$ is defined by the average value of the variational integral given by a solution with average slope α. Our main result is: A is differentiable at α if and only if the set $\mathscr{M}_\alpha$ is totally ordered (in the natural sense). In case that $\mathscr{M}_\alpha$ is not totally ordered, A is differentiable at α in some direction $\beta \in \mathbf{R}^n \setminus \{0\}$ if and only if β is orthogonal to the subspace defined by the rational dependency of α. Assuming that the i^{th} component of α is rational with denominator $s^i \in \mathbf{N}$ in lowest terms, we show: The difference of right- and left-sided derivative in the i^{th} standard unit direction is bounded by $\text{const} \cdot \frac{1}{s^i}$.

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1 Introduction

1.1 Setting of problem and related topics

The analytical and geometrical setting. The object of study may be viewed within two different settings – an analytical and a geometrical one. The specific topics are respectively:

A) the *minimal average action* of a variational problem on the $(n+1)$-dimensional torus,

B) the h-*stable norm* on the vector space $H_n(\mathrm{T}^{n+1}, \mathbf{R})$ of real homology classes of a Riemannian torus (T^{n+1}, g).

A) Within the setting A) the presented work investigates the differentiability properties of the minimal average action as a function of the rotation vector. We consider the variational problem lifted to the universal covering $\mathbf{R}^{n+1}$ of T^{n+1}. The lifted variational integrand $F = F(x,u,p) \in C^3(\mathbf{R}^{n+1} \times \mathbf{R}^n)$ is $\mathbf{Z}^{n+1}$-periodic in the first $(n+1)$ variables. We consider the **minimal average action** $A(\alpha)$ belonging to some rotation vector $\alpha \in \mathbf{R}^n$ which may be defined by

$$A(\alpha) = \inf_{u \in \mathscr{B}_\alpha} \liminf_{r \to \infty} \frac{1}{|B_r|} \int_{B_r} F(x,u,Du)\,dx \ ,$$

where $\mathscr{B}_\alpha \doteq \{ u \in C^3(\mathbf{R}^n) \, : \, \sup_{x \in \mathbf{R}^n} |u(x) - \alpha x| < \infty \}$ and where $B_r \subseteq \mathbf{R}^n$ is the ball of radius r and center 0.

The case $n = 1$ directly relates to monotone twist maps and thus to dynamical systems: in the phase space the time-1-map for extremals of the variational problem may be seen as an area preserving monotone twist map and vice versa [18]. In this 1-dimensional case, the differentiability of the minimal average action was first investigated by S. Aubry in the context of twist maps and their relations to physics [3]. For the corresponding discrete 1-dimensional variational problem he suggested in his words by 'physicists ideas' the following statement:

The minimal average action $A(\alpha)$ is differentiable at irrational rotation numbers $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ and generically not differentiable at rational rotation numbers $\alpha \in \mathbf{Q}$.

A rigorous proof of Aubry's statement was given by J. Mather [14]. The statement translates to the setting B) where for $n = 1$ the minimal solutions are interpreted as minimal geodesics on the torus T^2. In a more geometrical way V. Bangert proved the same result for a slightly generalized situation [7].

In case $n \geq 2$, the variational problem on T^{n+1} was first studied by J. Moser [17]. If $n \geq 2$, roughly speaking, the one-dimensional phenomenon occurs in every single direction. If e.g. $n = 2$ and $\alpha \in \mathbf{Q}^2$, the minimal average action A is generically non-differentiable at α in the direction e_1 and e_2 as well.

If $\alpha = (\alpha^1, \alpha^2) \in \mathbf{Q} \times \mathbf{R} \setminus \mathbf{Q}$ the minimal average action is differentiable at α in the direction e_2 because of the irrationality of the component α^2. The question of differentiability in the direction e_1, however, is more delicate than suggested by the 1-dimensional case. By the stability results of KAM-theory, the minimal average action is generically non-differentiable in the direction e_1 with $\alpha^1 \in \mathbf{Q}$ only if the second component $\alpha^2 \in \mathbf{R} \setminus \mathbf{Q}$ is well approximated by rationales [17]. Thus, the 1-dimensional situation does not fully apply.

In case $\alpha \in (\mathbf{R} \setminus \mathbf{Q})^2$ one has to distinguish whether α is rationally dependent or not. If there exists a relation $k^1\alpha^1 + k^2\alpha^2 \in \mathbf{Z}$ with $k = (k^1, k^2) \in \mathbf{Z}^2 \setminus \{0\}$, the minimal average action A at α may be non-differentiable in the direction k while it is always differentiable in the directions orthogonal to k. If $\alpha \in (\mathbf{R} \setminus \mathbf{Q})^2$ (more precisely $(-\alpha, 1)$) is rationally independent, i.e. if $\alpha k \notin \mathbf{Z}$ for all $k \in \mathbf{Z}^2 \setminus \{0\}$, the minimal average action is differentiable at α in *any* direction.

We generalize this result to higher dimensions and give a necessary and sufficient condition for the differentiability at an arbitrary rotation vector $\alpha \in \mathbf{R}^n$

in an arbitrary direction $\beta \in \mathbf{R}^n \setminus \{0\}$. A discussion of the main result is given in Sect. 3.

B) The notion of stable norm in setting B) goes back to a general construction by H. Federer. In a slight difference the **h-stable norm** which is related to homotopically instead of homologically area minimizing surfaces. If $\hat{\gamma} \in [T^2, T^3]$ is a homotopy class of smooth mappings $f : T^2 \to T^3$ we define $\|\hat{\gamma}\|_h \doteq \inf\{A(f) : f \in \hat{\gamma}\}$ where $A(f)$ is the area of f with respect to the Riemannian metric on T^3. Via isomorphism $[T^2, T^3] \simeq H_2(T^3, \mathbf{Z})$ one may continue $\|.\|_h$ homogeneously to a norm on $H_2(T^3, \mathbf{R})$. By the Poincaré duality we identify $H_2(T^3, \mathbf{R}) \simeq \mathbf{R}^3$. For details see [22].

Within the geometrical setting the results give an answer to a question of the type:

Which convex bodies in $\mathbf{R}^3$ may be realized by the unit ball of the h-stable norm on $H_2(T^3, \mathbf{R})$?

Recently this question was posed by Yu. Burago for the stable norm on $H_1(M, \mathbf{R})$, where M is a compact Riemannian manifold [10]. One is primarily interested in the differentiability of such a convex body. Up to now, by the result of Mather and Bangert, only the case $H_1(T^2, \mathbf{R})$ is known. Translated to the geometrical setting our result gives the following answer for the case $H_2(T^3, \mathbf{R})$:

Let B_h denote the unit ball of the h-*stable norm, ∂B_h its boundary. For $\gamma \in \mathbf{R}^3 \setminus \{0\}$ let $0 \le r \le 2$ be the degree of rational dependency of γ, i.e. the minimal number of linearly independent vectors $k \in \mathbf{Z}^3 \setminus \{0\}$ with $k\gamma = 0$. Then the tangent cone to ∂B_h at the point $\partial B_h \cap \mathbf{R}^+\gamma$, $\gamma \in H_2(T^3, \mathbf{R}) \setminus \{0\}$, contains at most $2 - r$ linearly independent straight lines. If the set $\mathcal{M}_\gamma$ of non-selfintersecting minimal surfaces on T^3 representing γ is not totally ordered the tangent cone contains* exactly *$2 - r$ linearly independent straight lines.*

According to [6], for large perturbations of the metric on T^3, the set $\mathcal{M}_\gamma$ has gaps for all $\gamma \in H_2(T^3, \mathbf{R}) \setminus \{0\}$. Together with the results in [8] this implies that $\mathcal{M}_\gamma$ is not totally ordered if the degree of rational dependency r of γ is > 0. Thus, the answer to the question above is quite striking: In the sense of measure the differentiability properties of ∂B_h may be as bad as it is possible for a convex body.

The physical interpretations. Besides its own mathematical interest, the study of the minimal average action is motivated from solid state physics. Corresponding to the analytical and geometrical setting, the results contribute to the problem of

A') phase locking in the multidimensional Frenkel-Kontorova model,

B') equilibrium form for idealized crystals.

A') The multidimensional Frenkel-Kontorova model may be viewed as a discrete version of the considered variational problem on the torus (see e.g. [9, 22]). In a physical language, the Frenkel-Kontorova model describes ground states of a grid of particles in a periodic potential with nearest neighbor interactions. Although the model is discrete, the methods of the variational problem directly carry over.

F. Vallet [23] calculated explicitly for a 2-dimensional example of a Frenkel-Kontorova-model the minimal average action $A(\alpha)$ and its derivative with respect to the rotation vector α. The non-differentiability of the minimal average action is shown to cause a so-called **phase-locking** of the physical system. The phase is identified with the rotation vector $\alpha \in \mathbf{R}^n$ and depends on a system parameter $\mu \in \mathbf{R}^n$. Given $\mu \in \mathbf{R}^n$ one looks for the unique $\alpha(\mu)$ such that $A(\tilde{\alpha}) - \mu\tilde{\alpha}$ is minimal for $\tilde{\alpha} = \alpha(\mu)$. The phase-function $\alpha(\mu)$ defined this way is locally constant if and only if A is nondifferentiable at $\alpha(\mu)$ in every direction. The (convex) region in the parameter space $\{\mu \in \mathbf{R}^n\}$ at which the phase $\alpha(\mu)$ is locked onto α_o is identified by the subdifferential of A at α_o. By our results, the function $\alpha(\mu)$ typically defines a (n-dimensional) **devil's staircase** while the family of subdifferentials typically defines a fractal structure on $\{\mu \in \mathbf{R}^n\}$. For details see [22].

B') The differentiability results of the h-stable norm has its consequences for the so called equilibrium form of crystals. In 1878 Gibbs proposed to consider a crystal as a body which minimizes its surface energy under the constraint of constant volume. The specific surface energy of a crystal face is assumed to depend only on the orientation of the face. Given the specific surface energy ϕ, the resulting isoperimetric problem is solved by the dual unit ball with respect to ϕ and thus may be constructed explicitly.

However, the problem is to determine the specific surface energy ϕ. Landau and Herring 1951 investigated an idealized crystal by considering the translationally invariant lattice structure only and therefrom deduced the typical properties of ϕ. Their model gives a first coherent motivation of Sohnke's famous **reciprocity law** according to which the size of a crystal face is indirectly proportional to its Miller indices [12].

Regarding the $\mathbf{Z}^3$-periodic crystal lattice as a periodic perturbation of the metric in $\mathbf{R}^3$, the specific surface energy ϕ may be interpreted as the h- stable norm $\|.\|_h$ on $H_2(T^3, \mathbf{R})$ with respect to the projected metric on T^3. Indeed, the corresponding results on the h-stable norm confirm in an astonishing way Landau's and Herring's heuristical statements on the properties of the specific surface energy and establish Sohnkes's reciprocity law (compare Theorem 3).

The differentiability properties of the specific surface energy ϕ with respect to the surface orientation determines the face-shape of the crystal. Each non-differentiability point of ϕ causes a face in the dual unit ball of ϕ and thus a face of the crystal. The size of the crystal face is proportional to the size of discontinuity in the derivative of ϕ. Thus, the crystal typically has a face normal to each rational direction and the size of the faces decrease if their normal directions becomes 'more irrational', i.e. if their Miller indices increase (see [22]).

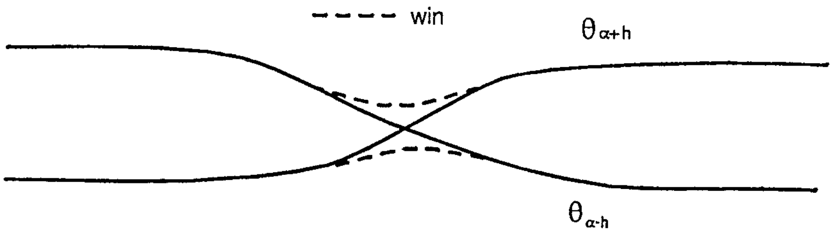

Fig. 1. $\max(\theta_{\alpha+h}, \theta_{\alpha-h})$ and $\min(\theta_{\alpha+h}, \theta_{\alpha-h})$ are not minimal because of the cusp upwards and downwards, respectively. Thus, the variational integral over $\theta_{\alpha+h}$ and $\theta_{\alpha-h}$ is larger than the variational integral over the short-cuts given by the dotted lines. This shows $(\frac{d}{d\alpha+} - \frac{d}{d\alpha-})A(\alpha) > 0$

1.2 A heuristic argument for the non-differentiability

We restrict to the case $n = 1$. There is an apparent difference between α rational and α irrational: If $\alpha \in \mathbf{R}\setminus\mathbf{Q}$, the set $\mathscr{M}_\alpha$ is totally ordered. On the contrary, if $\alpha \in \mathbf{Q}$, there are minimal solutions $\theta_{\alpha+h}$ and $\theta_{\alpha-h}$ in $\mathscr{M}_\alpha$ which intersect, i.e. $\mathscr{M}_\alpha$ is not totally ordered. The total ordering implies the differentiability of A at α while the intersection property at rational α implies the non-differentiability.

We give a rough idea of why in general A is not differentiable at $\alpha \in \mathbf{Q}$ as soon as a situation occurs of the type depicted in Fig. 1 for $\alpha = 0$ (i.e. as soon as $\mathscr{M}_\alpha$ does no more define a foliation).

For fixed α the minimal average action may be normalized by $A(\alpha) = 0$. If $h > 0$ is infinitesimally small one obtains for the difference of left- and right-sided derivative

$$\left(\frac{d}{d\alpha+} - \frac{d}{d\alpha-}\right)A(\alpha) = \frac{1}{h}(A(\alpha+h) + A(\alpha-h)) .$$

Since A is convex, the one-sided derivatives exist and the difference above is ≥ 0. Setting $2T = \frac{1}{h}$, we get by definition of A

$$\left(\frac{d}{d\alpha+} - \frac{d}{d\alpha-}\right)A(\alpha) = \int_{-T}^{T} F(t, \theta_{\alpha+h}, \dot\theta_{\alpha+h})\,dt + \int_{-T}^{T} F(t, \theta_{\alpha-h}, \dot\theta_{\alpha-h})\,dt .$$

We used that $\theta_{\alpha\pm h} = \lim_{\varepsilon\to 0}\theta_{\alpha\pm\varepsilon}$. The following reasoning shows that the right-hand side above is > 0.

First replace the functions $\theta_{\alpha+h}$ and $\theta_{\alpha-h}$ by $\max(\theta_{\alpha+h}, \theta_{\alpha-h})$ and $\min(\theta_{\alpha+h}, \theta_{\alpha-h})$ respectively. Of course, this does not change the sum of the total action from $-T$ to T. Now, the action of $\max(\theta_{\alpha+h}, \theta_{\alpha-h})$ and $\min(\theta_{\alpha+h}, \theta_{\alpha-h})$ may be reduced in the common edge by 'rounding' it (see Fig. 1).

Thus, $\max(\theta_{\alpha+h}, \theta_{\alpha-h})$ and $\min(\theta_{\alpha+h}, \theta_{\alpha-h})$ are surely not the 'cheapest ways' to connect their points at infinity. By minimality, the cheapest ways with the same asymptoticities are realized by $\theta_\alpha + 1$ and θ_α. But the total action of $\theta_\alpha + 1$ and θ_α from $-T$ to T, $T = \frac{1}{2h}$, is equal to 0 by the normalization $A(\alpha) = 0$ and the periodicity of θ_α. Therefore, the total action of $\max(\theta_{\alpha+h}, \theta_{\alpha-h})$ and $\min(\theta_{\alpha+h}, \theta_{\alpha-h})$ including an unrounded edge has to be > 0. Reordering again the integrals we have shown

$$\int_{-T}^{T} F(t, \theta_{\alpha+h}, \dot{\theta}_{\alpha+h})\, dt + \int_{-T}^{T} F(t, \theta_{\alpha-h}, \dot{\theta}_{\alpha-h})\, dt > 0 .$$

This establishes the non-differentiability of A at $\alpha \in \mathbf{Q}$ with respect to a generic integrand F.

If $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ the absence of an intersecting point in $\mathscr{M}_\alpha$ implies

$$\left(\frac{d}{d\alpha+} - \frac{d}{d\alpha-} \right) A(\alpha) = 0$$

with respect to any integrand F.

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2 The variational problem

2.1 The minimal average action

We consider a variational problem on $\mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$ of the form

$$\int_{\Omega \subseteq \mathbf{R}^n} F(x, u, Du)\, dx, \qquad D = \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right) ,$$

where F is defined on $\mathrm{T}^{n+1} \times \mathbf{R}^n$ and $u : \mathbf{R}^n \to \mathbf{R}$. The conditions on the integrand F are

$(F1)$ $\quad F = F(x, u, p) \in C^{2,\varepsilon}(\mathrm{T}^{n+1} \times \mathbf{R}^n)$,
in particular F has period 1 in $x^1, \dots, x^n, u$.

$(F2)$ $\quad$ There exists $\delta \in (0, 1]$ such that

$$\delta \|\xi\|^2 \le \sum_{\mu,\nu}^{n} F_{p^\mu p^\nu} \xi^\mu \xi^\nu \le \delta^{-1} \|\xi\|^2 \qquad \forall \xi \in \mathbf{R}^n ,$$

i.e. F satisfies the Legendre condition.

$(F3)$ $\quad$ There is $c_0 > 0$ such that

$$|F_{pu}| + |F_{px}| \le c_0(1 + \|p\|)$$
$$|F_{uu}| + |F_{ux}| + |F_{xx}| \le c_0(1 + \|p\|^2) .$$

A function $u \in W^{1,2}_{loc}(\mathbf{R}^n)$ is called a **minimal solution** if for all compactly supported $\phi \in W^{1,2}_{comp}(\mathbf{R}^n)$

$$\int_{supp\,\phi} (F(x, u + \phi, D(u + \phi)) - F(x, u, Du))\, dx \ge 0 .$$

The regularity theory asserts that minimal solutions again have the same smoothness as F, i.e. in our case $u \in C^{2,\varepsilon}(\mathbf{R}^n)$.

The typical example is of the form

$$F = \frac{1}{2}\|p\|^2 + V(x,u), \quad V \in C^{2,\varepsilon}(T^{n+1}),$$

where $V(x,u)$ is interpreted as a $\mathbf{Z}^{n+1}$-periodic potential on $\mathbf{R}^{n+1}$. For the **Dirichlet integrand** $F = \frac{1}{2}\|p\|^2$, the minimal solutions are exactly the harmonic functions.

On $C^0(\mathbf{R}^n)$ we define the $\mathbf{Z}^{n+1}$-action T as follows: If $\overline{k} = (k, k^{n+1}) \in \mathbf{Z}^{n+1}$ with $k = (k^1, \dots, k^n) \in \mathbf{Z}^n$ and if $u \in C^0(\mathbf{R}^n)$ set

$$T_{\overline{k}}u(x) \doteq u(x-k) + k^{n+1}.$$

u is said to be **non-selfintersecting** (on T^{n+1}) if the T-orbit of u is totally ordered, i.e. if for all $\overline{k} \in \mathbf{Z}^{n+1}$ either $T_{\overline{k}}u > u$ or $T_{\overline{k}}u = u$ or $T_{\overline{k}}u < u$.

One shows that to every non-selfintersecting $u \in C^0(\mathbf{R}^n)$ there is a so called **rotation vector** $\alpha \in \mathbf{R}^n$ such that

$$\sup |u(x) - \alpha x| < \infty . \tag{1}$$

The set of all non-selfintersecting minimal solutions corresponding to a fixed rotation vector α is denoted by $\mathscr{M}_\alpha$. According to [15, thm 5.6] $\mathscr{M}_\alpha \neq \emptyset$ for all $\alpha \in \mathbf{R}^n$. We say that $\mathscr{M}_\alpha$ defines a **foliation** of $\mathbf{R}^{n+1}$ if to every $\overline{x} = (x, x^{n+1}) \in \mathbf{R}^{n+1}$ with $x \in \mathbf{R}^n$ there is exactly one $u \in \mathscr{M}_\alpha$ with $u(x) = x^{n+1}$. For the Dirichlet-integrand $\mathscr{M}_\alpha$ foliates $\mathbf{R}^{n+1}$ by affine hyperplanes of slope α.

If a minimal solution u satisfies (1) with some $\alpha \in \mathbf{R}^n$, we define the **minimal average action** of u or α, respectively, as

$$A(\alpha) \doteq \lim_{r\to\infty} \frac{1}{|B_r|} \int_{B_r} F(x,u,Du)\,dx$$

where $B_r \subseteq \mathbf{R}^n$ is a ball of radius r and $|B_r|$ its volume. In [20, Satz 3.4] it is shown that this limit indeed exists and does not depend on the minimal solution u satisfying (1). Moreover, $A(\alpha)$ is shown to be a strictly convex function on $\mathbf{R}^n$ [20, Kor. 4.2].

For the statement of the differentiability properties of $A(\alpha)$ one directly may skip to Sect. 3.1.

2.2 The set of non-selfintersecting minimal solutions

Our proof of the differentiability properties of $A(\alpha)$ relies on the uniform smoothness of the corresponding minimal solutions. For $\mathscr{A} \geq 0$ put $\mathscr{M}_{\mathscr{A}} \doteq \bigcup_{|\alpha|\leq\mathscr{A}} \mathscr{M}_\alpha$. According to the fundamental work of Moser [15, thm 3.1, 4.3] we have

Lemma 1 *For every $\mathscr{A} \geq 0$ there is a constant $c_1 > 0$ depending only on δ and c_0 such that for all $u \in \mathscr{M}_{\mathscr{A}}$*

$$\|Du\|_{C^\epsilon} \leq c_1 \ .$$

If moreover $u, v \in \mathscr{M}_{\mathscr{A}}$ and $u < v$ in a ball B, then

$$\|Dv(x) - Du(x)\| \leq c_1 |v(x) - u(x)| \quad \textit{for all } x \in \frac{1}{2}B \ .$$

From the lemma one deduces that for $\mathscr{A} \geq 0$ the sets $\{u \in \mathscr{M}_{\mathscr{A}} : |u(0)| \leq \text{const}\} \subset \mathscr{M}_{\mathscr{A}}$ is compact with respect to the C^1-topology on compact sets [15, Cor. 3.3].

In the following we describe the structure of the set $\mathscr{M}_\alpha$ as it is established by Bangert in [5]. To any $\alpha \in \mathbf{R}^n$ we associate the unit normal vector $\overline{a}_1 \doteq \frac{(-\alpha,1)}{\|(-\alpha,1)\|} \in S^n$ of the hyperplane $x^{n+1} = \alpha x$. We define the lattice

$$\overline{\Gamma}_\alpha \doteq \mathbf{Z}^{n+1} \cap \langle \overline{a}_1 \rangle^\perp = \{\overline{k} \in \mathbf{Z}^{n+1} : \overline{k}\overline{a}_1 = 0\} \ .$$

Let Γ_α be the projection of $\overline{\Gamma}_\alpha$ to $\mathbf{Z}^n$ neglecting the last component. Thus $\Gamma_\alpha = \{k \in \mathbf{Z}^n : \alpha k \in \mathbf{Z}\}$. By $\mathscr{M}(\overline{a}_1) \subseteq \mathscr{M}_\alpha$ we denote the set of **maximally periodic** $u \in \mathscr{M}_\alpha$, i.e.

$$\mathscr{M}(\overline{a}_1) \doteq \{u \in \mathscr{M}_\alpha : T_{\overline{k}} u = u \ \forall \overline{k} \in \overline{\Gamma}_\alpha\} \ .$$

This set is closed and totally ordered. The graphs of its elements therefore define either a foliation of $\mathbf{R}^{n+1}$ or a **lamination**, i.e. a foliation probably with gaps. In the case of a foliation one has $\mathscr{M}(\overline{a}_1) = \mathscr{M}_\alpha$.

We say that $\overline{\alpha} = (-\alpha, 1)$ is **rationally dependent** iff $\overline{\Gamma}_\alpha \neq \{0\}$, i.e. iff there exists $\overline{k} \in \mathbf{Z}^{n+1} \setminus \{0\}$ such that $\overline{\alpha}\overline{k} = 0$. This is equivalent to $\Gamma_\alpha \neq \{0\}$. If $\overline{\alpha}$ is rationally dependent, we will see that the occurrence of gaps in (the union of graphs of) $\mathscr{M}(\overline{a}_1)$ is responsible for the non-differentiability of $A(\alpha)$.

Now suppose $(-\alpha, 1)$ is rationally dependent and suppose $\mathscr{M}(\overline{a}_1)$ gives rise to a lamination with gaps. Set

$$V_\alpha \doteq \text{span}_{\mathbf{R}} \Gamma_\alpha = \text{span}_{\mathbf{R}}\{k \in \mathbf{Z}^n : \alpha k \in \mathbf{Z}\} \subseteq \mathbf{R}^n$$

and choose any direction $\beta \in V_\alpha \cap S^{n-1}$. According to [5, (7.1)] there are minimal solutions $u \in \mathscr{M}_\alpha$ the graphs of which lie within the gaps of $\mathscr{M}(\overline{a}_1)$ and which have the following asymptotic behavior:

In the direction β, u is asymptotic to some $u^+ \in \mathscr{M}(\overline{a}_1)$ while in the opposite direction $-\beta$, u is asymptotic to some $u^- \in \mathscr{M}(\overline{a}_1)$. One has $u^- < u < u^+$ and u^-, u^+ are neighboring, i.e. there is no other minimal solution in $\mathscr{M}(\overline{a}_1)$ lying between u^- and u^+. To formulate the asymptotic behavior of u more precisely we have to consider translates of graph u in the directions $\overline{k}_j = (k_j, k_j^{n+1}) \in \mathbf{Z}^{n+1}$ with $\lim_{j\to\infty} k_j \beta = \pm\infty$. In order to express this last condition in terms of $\overline{k}_j$ instead of k_j we replace $\beta \in V_\alpha \cap S^{n-1}$ by the unique vector $\overline{\beta} \in \overline{V}_\alpha \doteq \text{span}_{\mathbf{R}} \overline{\Gamma}_\alpha$ with the property that

$$\overline{k\beta} = k\beta \quad \text{for all } \overline{k} = (k, k^{n+1}) \in \overline{\Gamma}_\alpha .$$

For consistency with the terminology in [5, (4.4)] we set

$$\overline{a}_2 \doteq -\frac{\overline{\beta}}{\|\overline{\beta}\|} \in \overline{V}_\alpha \cap S^n .$$

Note that $\overline{a}_2 \perp \overline{a}_1$. We introduce the sublattices

$$\overline{\Gamma}_{\alpha,\beta} \doteq \overline{\Gamma}_\alpha \cap \langle \overline{a}_2 \rangle^\perp = \{\overline{k} \in \mathbf{Z}^{n+1} : \overline{k}\overline{a}_1 = \overline{k}\overline{a}_2 = 0\}$$

and $\Gamma_{\alpha,\beta} \doteq \pi(\overline{\Gamma}_{\alpha,\beta}) = \Gamma_\alpha \cap \langle \beta \rangle^\perp$, π being the projection of $\mathbf{Z}^{n+1}$ to $\mathbf{Z}^n$ forgetting the last component. With these definitions the set described above of all $u \in \mathcal{M}_\alpha$ asymptotic in the directions $\pm\beta$ to neighboring $u^- < u^+ \in \mathcal{M}(\overline{a}_1)$ may be written as

$$\mathcal{M}(\overline{a}_1, \overline{a}_2) \doteq \{u \in \mathcal{M}_\alpha : \; T_{\overline{k}} u = u \; \forall \overline{k} \in \overline{\Gamma}_{\alpha,\beta} \text{ and } \; T_{\overline{k}} u > u \; \forall \overline{k} \in \overline{\Gamma}_\alpha \text{ with } \overline{k}\overline{a}_2 > 0\} .$$

If $\mathcal{M}(\overline{a}_1)$ does not define a foliation then for each $\beta \in V_\alpha \cap S^{n-1}$ one has $\mathcal{M}(\overline{a}_1, \overline{a}_2) \neq \emptyset$. The asymptotic behavior of $u \in \mathcal{M}(\overline{a}_1, \overline{a}_2)$ in the directions $\pm\beta$ translates to

$$\lim_{j\to\infty} T_{\overline{k}_j} u = u^\pm \quad \text{if } \overline{k}_j \in \overline{\Gamma}_\alpha \text{ with } \lim_{j\to\infty} \overline{k}_j \overline{a}_2 = \pm\infty .$$

The limit $\lim T_{\overline{k}_j} u = u^\pm$ is understood with respect to the C^1-topology on compact sets. Because of their asymptotic behavior, we say that a minimal solutions $u \in \mathcal{M}(\overline{a}_1, \overline{a}_2)$ is **heteroclinic** in the direction β with respect to the $\mathbf{Z}^{n+1}$-action T and periodic (mod $\mathbf{Z}$) in the directions determined by $\Gamma_{\alpha,\beta}$. The fact that $\mathcal{M}(\overline{a}_1) \cup \mathcal{M}(\overline{a}_1, \overline{a}_2)$ is totally ordered [5, (7.4)] will be essential for our investigation of the derivative of $A(\alpha)$.

Finally we mention that for $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ one has the special class of all recurrent minimal solutions $\mathcal{M}_\alpha^{rec}$ lying within the set $\mathcal{M}(\overline{a}_1)$. A minimal solution $u \in \mathcal{M}_\alpha$ is said to be **recurrent** if

$$u = \inf\{T_{\overline{k}} u : \overline{k}\overline{a}_1 > 0\} \qquad \text{or} \qquad u = \sup\{T_{\overline{k}} u : \overline{k}\overline{a}_1 < 0\} .$$

According to [4, (5.2)] the set $\mathcal{M}_\alpha^{rec} \subseteq \mathcal{M}(\overline{a}_1)$ may be characterized as the unique minimal set of the $\mathbf{Z}^{n+1}$-action T on $\mathcal{M}(\overline{a}_1)$. The set $\{u(0) : u \in \mathcal{M}_\alpha^{rec}\} \subseteq \mathbf{R}$ is either homeomorphic to $\mathbf{R}$ or to a periodic Cantor set depending on whether $\mathcal{M}_\alpha^{rec}$ gives rise to a foliation or not. In the case of a foliation one has $\mathcal{M}_\alpha^{rec} = \mathcal{M}(\overline{a}_1)$ because of the total order of $\mathcal{M}(\overline{a}_1)$. According to Bangert's classification of the minimal solution without selfintersection, in this case even $\mathcal{M}_\alpha^{rec} = \mathcal{M}_\alpha$. If on the other hand $\mathcal{M}_\alpha^{rec}$ exhibits gaps one may have strict inclusions. If $\alpha \in \mathbf{Q}^n$ we have $\mathcal{M}(\overline{a}_1) = \mathcal{M}_\alpha^{rec}$ and this will also be denoted by $\mathcal{M}_\alpha^{per}$.

3 Differentiability results for the minimal average action

In this chapter we state and discuss the main results. Since the minimal average action is convex, the (one-sided) directional derivative $D_\beta A(\alpha)$ in any direction $\beta \in S^{n-1}$ exists and $(D_\beta + D_{-\beta})A(\alpha) \geq 0$ for all $\alpha \in \mathbf{R}^n$. Again we abbreviate $S^{n-1} = \{x \in \mathbf{R}^n : \|x\| = 1\}$. If equality holds at some $\alpha \in \mathbf{R}^n$ for n linearly independent directions $\beta \in S^{n-1}$, then it actually holds for every $\beta \in S^{n-1}$, i.e. A is differentiable at α. By convexity, A is differentiable almost everywhere. Moreover, restricted to the set $D \subseteq \mathbf{R}^n$ where A is differentiable, $A|_D$ is even *continuously* differentiable [19, thm 25.5].

Although the convexity property determines the differentiability of a function to some extent, its s-singular points, $0 \leq s \leq n$, may be distributed in a very complicated manner. The theorems in the first section describe what is possible for the minimal average action A. The second section gives an explicit formula for the directional derivatives of A.

3.1 Qualitative behavior: differentiability in a particular direction

Theorem 1 *If $\mathcal{M}_\alpha$ gives rise to a foliation of $\mathbf{R}^{n+1}$ then the minimal average action A is differentiable at the point α.*

Conversely, it is not so that A is non-differentiable at α as soon as $\mathcal{M}_\alpha$ gives rise to gaps. A further necessary condition is that $\overline{\alpha} = (-\alpha, 1)$ is rational dependent, i.e. that $V_\alpha \neq \{0\}$. In order to decide whether or not the left- and right-sided derivatives of A in some direction $\beta \in S^{n-1}$ coincide one has to check whether β has a nontrivial component in $V_\alpha = \text{span}_{\mathbf{R}}\{k \in \mathbf{Z}^n : \alpha k \in \mathbf{Z}\}$.

By $V_\alpha^\perp$ we denote the orthogonal complement of V_α in $\mathbf{R}^n$.

Theorem 2 *Let $\alpha \in \mathbf{R}^n$, $\beta \in \mathbf{R}^n \setminus \{0\}$ and suppose $\mathcal{M}_\alpha$ does not give rise to a foliation. Then*

$$(D_\beta + D_{-\beta})A(\alpha) \begin{cases} = 0 & \text{if } \beta \in V_\alpha^\perp \\ > 0 & \text{else} \end{cases} .$$

If in particular $\overline{\alpha}$ is rationally independent, all partial derivatives exist and A is differentiable at α (irrespective of whether $\mathcal{M}_\alpha$ defines a foliation or not).

If $n = 1$ the minimal average action A is differentiable at any $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ while at $\alpha \in \mathbf{Q}$ it is differentiable if and only if $\mathcal{M}_\alpha$ defines a foliation. The analogous result was found by J. Mather [14] and V. Bangert [7, (5.3)] for a 1-dimensional discrete variational problem. While our proof for the rational case $\alpha \in \mathbf{Q}^n$, $n = 1$, is based on the same idea as Mather's work, our deduction of the irrational case $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ is simpler. In particular, we do not need to estimate the convergence of the difference quotient at $\alpha \in \mathbf{Q}$ quantitatively. The case $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ is obtained by a limit process. Mather's argument is replaced by the fact that the corresponding minimal laminations converge. In the case $n = 1$ it is enough to consider the subset $\mathcal{M}_\alpha^{rec} \subseteq \mathcal{M}_\alpha$ instead of $\mathcal{M}_\alpha$ itself. The reason

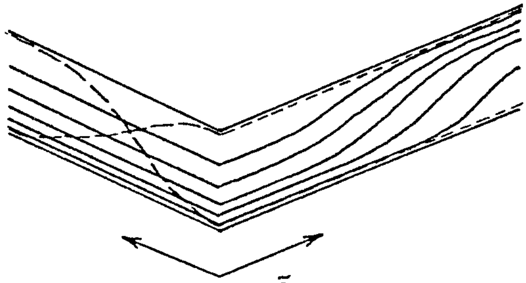

Fig. 2. A situation not arising in the 1-dimensional case. One has $(D_\beta + D_{-\beta})A(\alpha) > 0$ since $\mathcal{M}_\alpha$ itself does not give rise to a foliation although a genuine subset of $\mathcal{M}_\alpha$ foliates $\mathbf{R}^3$ even periodically in the direction β (fat lines)

is that for $n = 1$ the set $\mathcal{M}_\alpha^{rec}$ ($= \mathcal{M}(\overline{a}_1)$ for $n = 1$) defines a foliation of $\mathbf{R}^2$ if and only if $\mathcal{M}_\alpha$ does. For $n \geq 2$ this seems no longer to be guaranteed. As the following two situation illustrates, the question whether left- and right sided derivative coincide is more subtle for $n \geq 2$:

Suppose that for $\alpha \in \mathbf{R}^n$ with $\dim_R V_\alpha \geq 2$ some subset of $\mathcal{M}_\alpha$ is foliating $\mathbf{R}^{n+1}$. Assume that the foliation is periodic in a rational direction $\beta \in V_\alpha$. Guided by the 1-dimensional case one could expect that this ensures $(D_\beta + D_{-\beta})A(\alpha) = 0$. However, the leaves of the foliation may be heteroclinic in a further direction $\tilde{\beta} \in V_\alpha$ orthogonal to β (i.e. lie in $\mathcal{M}(\overline{a}_1, \tilde{\overline{a}}_2)$). This situation occurs e.g. for the integrand $F = \frac{1}{2}|p|^2 - \cos(2\pi D_{\tilde{\beta}} u)$ at $\alpha = (0,0) \in \mathbf{R}^2$ with $\beta = (1,0)$, $\tilde{\beta} = (0,1)$. In this case $\mathcal{M}_\alpha$ has to contain necessarily solutions which are heteroclinic in the direction $-\tilde{\beta}$ (i.e. lie in $\mathcal{M}(\overline{a}_1, -\tilde{\overline{a}}_2)$) and thus intersect some leaves of the foliation. Theorem 2 implies $(D_\beta + D_{-\beta})A(\alpha) > 0$.

The example shows that the existence of a periodic minimal foliation to α in some direction $\beta \in V_\alpha \setminus \{0\}$ does not imply $(D_\beta + D_{-\beta})A(\alpha) = 0$. The crucial notion to decide whether A is differentiable at $\alpha \in \mathbf{R}^n$ or not is the one of 'total ordering' of $\mathcal{M}_\alpha$:

Corollary 1 *The minimal average action A is differentiable at $\alpha \in \mathbf{R}^n$ if and only if $\mathcal{M}_\alpha$ is totally ordered. In case that $\mathcal{M}_\alpha$ is not totally ordered, A is differentiable at α in the direction $\beta \in \mathbf{Q}^n \setminus \{0\}$ if and only if $\alpha\beta \in \mathbf{R} \setminus \mathbf{Q}$.*

There is a further difference to the 1-dimensional case. If $n = 1$ the minimal average action for generic integrands F is not differentiable at rational $\alpha \in \mathbf{Q}$ while it is differentiable at irrational $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ for any integrand F. If $n \geq 2$ the action $A(\alpha)$ is still generically not differentiable at $\alpha \in \mathbf{Q}^n$, while it is always differentiable only at rationally independent $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$. If $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ is rationally dependent the statement is more complicated. (For convenience, we say that α is rationally independent if $\overline{\alpha}$ is.) If e.g. $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ is 'not too near' to $\mathbf{Q}^n$, a smooth foliation of $\mathbf{R}^{n+1}$ by $\mathcal{M}_\alpha$ will survive as foliation under small perturbations of the integrand F [17], although it may lose its smoothness. Thus, at such α the set $\mathcal{M}_\alpha$ has not generically gaps and according to our theorem, the minimal average action cannot be generically nondifferentiable. This can even happen if all but one component of α are rational. Conversely, there exist $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ such that a foliation to α desintegrates under arbitrarily small perturbations. Therefore, $A(\alpha)$ cannot be generically differentiable at such α.

In any case, for sufficiently large perturbations of the variational integrand F one expects that $\mathcal{M}_\alpha$ has gaps for every α in a given compact set $K \subseteq \mathbf{R}^n$. If F is the Dirichlet integrand such a result is proven in [6]. Assuming this situation, Theorem 2 says that the differentiability behavior of A is in some sense as bad as it can be for a convex function.

For an arbitrary convex function $f : K \subseteq \mathbf{R}^n \to \mathbf{R}$ a point x in its domain K is said to be **s-singular** if there exist not more than s linearly independent directions in which left- and right-sided derivatives at x coincide. Note that $x \in K$ is n-singular if and only if f is differentiable at x. The general theorem asserts that their are not too many s-singular points [1, (3.1)] :

(2.0) *The set of s-singular points of a convex function $f : K \to \mathbf{R}$, $K \subseteq \mathbf{R}^n$ compact, is the union of countably many compact sets of finite s-dimensional Hausdorff measure ($s = 0, \dots, n$).*

By the **degree** of rational dependency of $\overline{\alpha}$ and α, respectively, we denote the dimension of $V_\alpha = \mathrm{span}_{\mathbf{R}}\{k \in \mathbf{Z}^n : \alpha k \in \mathbf{Z}\}$. According to Theorem 2 a point $\alpha \in \mathbf{R}^2$ with rational dependency of degree r is a $(n-r)$-singular point if $\mathcal{M}_\alpha$ has gaps. If $\mathcal{M}_\alpha$ has gaps for any α in a compact set $K \subseteq \mathbf{R}^n$, the points $\alpha \in K$ are $(n - \dim V_\alpha)$-singular points of $A|_K$. The differentiability of $A|_K$ cannot be worse: for each $s = 0, \dots, n$ one has to take a union of *infinitely* many compact sets of finite s-dimensional Hausdorff measure to get all s-singular points of $A|_K$.

On this background, the continuity properties which may be established for $D_\beta A(\alpha)$ with respect to α are not at all evident. We give a condition on the approximating sequence $\alpha' \to \alpha$ insuring the convergence of the corresponding derivatives $D_\beta A(\alpha')$ to $D_\beta A(\alpha)$. (See Theorem 4, Sect. 5.3.)

In view of Theorem 2 one wishes to have an upper bound for $(D_\beta + D_{-\beta})A(\alpha)$ in terms of α and β. Although one would expect stronger estimates, the following theorem is a first step in this direction.

Theorem 3 *Let $\alpha \in \mathbf{R}^n$ and suppose the rational components of α with respect to the standard basis $\{e_1, \dots, e_n\}$ have the form $\alpha^i = \frac{r^i}{s^i}$ with $r^i \in \mathbf{Z}$ and $s^i \in \mathbf{N}$ relatively prime. Let $K \subseteq \mathbf{R}^n$ be compact. There is a constant depending on F and K only such that for all $\alpha \in K \cap \mathbf{Q}^n$ and all $1 \leq i \leq n$ with $\alpha^i \in \mathbf{Q}$ one has*

$$0 \leq (D_{e_i} + D_{-e_i})A(\alpha) \leq \text{const}\ \frac{1}{s^i}\ .$$

The analogous statement is true for derivatives in any rational direction $\beta \in V_\alpha$ with $\|\beta\| = 1$. Note that the theorem does not allow to deduce the existence of the i-th partial derivative of $A(\alpha)$ with $\alpha^i \in \mathbf{R} \setminus \mathbf{Q}$ by a limit process since the corresponding continuity property is not guaranteed. According to Theorem 2 one has to require in addition $e_i \in V_\alpha^\perp$.

3.2 Quantitative behavior: an explicit formula

We give a formula for the derivative $D_\beta A(\alpha)$ in the direction β which holds for β lying in V_α as well as for β lying in $V_\alpha^\perp$.

Let $S_Q^{n-1} \doteq \{\frac{\gamma}{\|\gamma\|} \in S^{n-1} : \gamma \in \mathbf{Q}^n \setminus \{0\}\}$ be the set of all rational directions in $\mathbf{R}^n$. Given $\alpha \in \mathbf{R}^n$ choose $\beta \in (V_\alpha \cup V_\alpha^\perp) \cap S_Q^{n-1}$. If $\beta \in V_\alpha$ we associate to β the unique vector $\overline{a}_2 \in \overline{V}_\alpha \cap S^n$ as defined in Sect. 2. Recall $\overline{a}_1 = \frac{(-\alpha,1)}{\|(-\alpha,1)\|}$. In order to not have to distinguish between $\beta \in V_\alpha$ and $\beta \in V_\alpha^\perp$ we abbreviate

$$\mathscr{M}_{\alpha,\beta} \doteq \begin{cases} \mathscr{M}(\overline{a}_1) \cup \mathscr{M}(\overline{a}_1, \overline{a}_2) , & \text{if } \beta \in V_\alpha \\ \mathscr{M}(\overline{a}_1) , & \text{if } \beta \in V_\alpha^\perp . \end{cases}$$

Thus, $\mathscr{M}_{\alpha,\beta}$ consists of all $u \in \mathscr{M}_\alpha$ which are maximally periodic and, in case $\alpha\beta \in \mathbf{Q}$, moreover of all $u \in \mathscr{M}_\alpha$ which are heteroclinic in the direction β. If $n = 1$ and $\alpha \in \mathbf{Q}$ other notations for $\mathscr{M}_{\alpha,\beta}$ used in literature are $\mathscr{M}_{\alpha+}$ or $\mathscr{M}_\alpha \cup \mathscr{M}_\alpha^+$. If $n = 1$ and $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ one has $\mathscr{M}_{\alpha,\beta} = \mathscr{M}_\alpha^{rec}$.

The formulation of the main theorem requires a few additional definitions. Fix an arbitrary minimal solution $u_\alpha \in \mathscr{M}(\overline{a}_1)$. For $\Omega \subseteq \mathbf{R}^n$ let

$$\mathscr{F}_{\alpha,\beta}(\Omega) \doteq \bigcup \{ \text{ graph } u|_\Omega : u \in \mathscr{M}_{\alpha,\beta} \ , \ u_\alpha - 1 < u < u_\alpha \} \subseteq \mathbf{R}^{n+1}$$

be the corresponding part of the generalized foliation associated to $\mathscr{M}_{\alpha,\beta}$. We define the function

$$\Psi_{\alpha,\beta} : \mathscr{F}_{\alpha,\beta}(\mathbf{R}^n) \to \mathbf{R}^n \ , \qquad \Psi_{\alpha,\beta}(\overline{x}) = Du(x) \ ,$$

where $\overline{x} = (x, x^{n+1})$ and u is the unique solution in $\mathscr{M}_{\alpha,\beta}$ with $u(x) = x^{n+1}$. For functions $u < v \in \mathscr{M}_{\alpha,\beta}$ we denote the open set lying between their graphs by (u, v), i.e.

$$(u, v) \doteq \{(x, x^{n+1}) \in \mathbf{R}^{n+1} : u(x) < x^{n+1} < v(x), \ x \in \mathbf{R}^n\} \ .$$

By $\mathscr{G}_{\alpha,\beta}$ we denote the set of gaps of the generalized foliation $\mathscr{F}_{\alpha,\beta}(\mathbf{R}^n)$. More precisely,

$$\mathscr{G}_{\alpha,\beta} \doteq \{ G \subset \mathbf{R}^{n+1} : G \text{ is a connected component of } (u_\alpha - 1, u_\alpha) \backslash \mathscr{F}_{\alpha,\beta}(\mathbf{R}^n) \} \ .$$

For each gap $G \in \mathscr{G}_{\alpha,\beta}$ there are unique $u_G^- < u_G^+ \in \mathscr{M}_{\alpha,\beta}$ such that $G = (u_G^-, u_G^+)$. If $G \subset \mathbf{R}^{n+1}$ is a set of the form $G = (u_G^-, u_G^+)$ and $\Omega \subseteq \mathbf{R}^n$, we define

$$B_G(\Omega) \doteq \int_\Omega (F(x, u_G^-, Du_G^-) - F(x, u_G^+, Du_G^+))\, dx \ .$$

Without loss of generality we assume the standard situation where $V_\alpha = \text{span}_{\mathbf{R}}\{k \in \mathbf{Z}^n : k\alpha \in \mathbf{Z}\}$ is either trivial or spanned by the standard unit vectors $e_1, \dots, e_r$ with $1 \le r = \dim V_\alpha \le n$. In other words, the first r components of $\alpha = (\alpha^1, \dots, \alpha^r, \alpha^{r+1}, \dots, \alpha^n)$ are assumed to be rational and the last $n - r$ components are are assumed to be irrational as well as rationally independent. The general situation is obtained by a linear transformation.

Put $E_0 \doteq [0,1]^n$ and $\mathbf{R}^+ \doteq \{\lambda \in \mathbf{R} : \lambda > 0\}$. Set $E^+_{0,e_i} \doteq E_0 + \mathbf{R}^+ e_i$ and let E°_{0,e_i} be the face of E_0 orthogonal to e_i and containing $0 \in \mathbf{R}^n$.

Main Theorem *Suppose $\alpha \in \mathbf{R}^n$ with $V_\alpha = \mathbf{R}^r \times \{0\}^{n-r}$ for some $0 \le r \le n$. The (one-sided) directional derivative of A at α in any standard unit direction $e_i \in \mathbf{R}^n$, $1 \le i \le n$, has the form*

$$D_{e_i} A(\alpha) = \int_{\mathscr{F}_{\alpha,e_i}(E^\circ_{0,e_i})} F_{p^i}(\overline{x}, \Psi_{\alpha,e_i})\, dx^1 \overset{i}{\ldots} dx^{n+1} + \sum_{G \in \mathscr{G}_{\alpha,e_i}} B_G(E^+_{0,e_i}) \tag{2}$$

where the order of summation over $\mathscr{G}_{\alpha,e_i}$ corresponds to decreasing absolute values of $B_G(E^+_{0,e_i})$.

The disadvantage that for $n \ge 2$ and $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ one has to precise the order of summation is overcome with some expense in formula (8) below. An alternative way of integration in formula (2) is given by the remark in Sect. 4.1. First, we discuss special cases of (2).

If $n = 1$ and $\alpha \in \mathbf{Q}$, one shows that generically there exists up to translations exactly one periodic minimal solution in $\mathscr{M}_\alpha$. If $n = 1$ and $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, a corresponding statement exists for generic monotone twist maps. According to [13], for generic monotone twist maps and generic irrational rotation number $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, the Aubry-Mather set is hyperbolic. Fathi showed that in this case it has Hausdorff dimension 0, [11, thm 2.1]. By the correspondence between monotone twist maps and $\mathbf{Z}^2$-periodic variational problems [18] his statement translates to the set $\mathscr{F}_{\alpha,e_1}(0) = \{u(0) : u \in \mathscr{M}^{rec}_\alpha, u_\alpha - 1 \le u \le u_\alpha\}$.

Thus, for $n = 1$ and arbitrary $\alpha \in \mathbf{R}$ formula (2) generically reduces to

$$\frac{d}{d\alpha+} A(\alpha) = \sum_{G \in \mathscr{G}_{\alpha,e_1}} B_G(\mathbf{R}^+)\,, \tag{3}$$

where $\frac{d}{d\alpha+}$ denotes the right-sided derivative with respect to α.

For the 1-dimensional discrete variational problem a formula analogous to (2) is established. In [14] it corresponds to the expression for K^+. In case $\alpha \in \mathbf{Q}$ our formula (3) corresponds to (8.4a) in [2], where the derivative of the average action is calculated for the standard twist map in the anti-integrable case. Aubry not only shows that in his anti-integrable case the derivative jumps at all rational rotation numbers α but even that the variation of A vanishes on the set of all irrational α. However, if we do not assume to be near the anti-integrable case and we only know that $\mathscr{M}_\alpha$ exhibits gaps, similar questions remain unsettled even for $n = 1$.

While in (2) the first term cancels if $\bigcup G$ has full measure, the second term cancels if $\mathscr{M}_\alpha$ defines a foliation. In this case $\mathscr{M}_{\alpha,\beta}$ is equal to $\mathscr{M}_\alpha$ for any $\beta = e_i$, $1 \le i \le n$. Moreover, the function $\Psi_\alpha \doteq \Psi_{\alpha,\beta}$ is independent of β and may be considered as a function on $\overline{E}_0 \doteq [0,1]^{n+1}$ with $\Psi_\alpha(\overline{x}) = Du(x)$ for $u \in \mathscr{M}_\alpha$ with $u(x) = x^{n+1}$. From the main theorem we deduce

Corollary 2 *If $\mathcal{M}_\alpha$ gives rise to a foliation, the partial derivatives of A at α exist and*

$$\frac{\partial}{\partial \alpha^i} A(\alpha) = \int_{\overline{E}_0} F_{p^i}(\overline{x}, \Psi_\alpha)\, d\overline{x}\ , \qquad 1 \le i \le n\ . \tag{4}$$

Since the existence of all partial derivatives of a convex function at some point implies its differentiability at this point, Theorem 1 follows immediately, see [19, 25.2].

Proof of Corollary 2 If $\alpha \in \mathbf{R}^n$ and $\Omega \subseteq \mathbf{R}^n$ set

$$\mathcal{F}_\alpha(\Omega) \doteq \bigcup \{\text{ graph } u|_\Omega : u \in \mathcal{M}(\overline{a}_1),\ u_\alpha - 1 \le u \le u_\alpha \}\,.$$

We show that irrespective whether $\mathcal{M}_\alpha$ defines a foliation or not one has

$$\int_{\mathcal{F}_\alpha(E^\circ_{0,e_i})} F_{p^i}(\overline{x}, \Psi_\alpha)\, dx^1 \overset{i}{\dots} dx^{n+1} = \int_{\mathcal{F}_\alpha(E_0)} F_{p^i}(\overline{x}, \Psi_\alpha)\, d\overline{x}\ , \tag{5}$$

i.e. that the domain of integration $\mathcal{F}_\alpha(E_0) = \{\overline{x} = (x, u(x)) : x \in E_0, u \in \mathcal{M}(\overline{a}_1), u_\alpha - 1 \le u \le u_\alpha\}$ may be restricted to those $\overline{x}$ with $x^i = 0$. (However, (5) is not true in general if on both sides $\mathcal{F}_\alpha$ is replaced by $\mathcal{F}_{\alpha,e_i}$.)

Let $U^+_\alpha(x,\theta)$ be the function from [15] conjugating the lamination $\mathcal{M}(\overline{a}_1)$ to the affine foliation $\{\alpha x + c : c \in \mathbf{R}\}$ of $\mathbf{R}^{n+1}$. Thus, for any $c \in \mathbf{R}$ the function $x \to U^+_\alpha(x, \alpha x + c)$ defines a minimal solution in $\mathcal{M}(\overline{a}_1)$ and up to a countable number all solutions in $\mathcal{M}(\overline{a}_1)$ have such a representation. Moreover, $U^+_\alpha(x,\theta)$ is monotone and upper semi-continuous in θ and satisfies the periodicity properties

$$U^+_\alpha(x + e_i, \theta) = U^+_\alpha(x,\theta)\,, \quad U^+_\alpha(x, \theta + 1) = U^+_\alpha(x,\theta) + 1\,,$$

with $1 \le i \le n$, see [15, thm 6.3]. If $\bar{D} = (\bar{D}_1, \dots, \bar{D}_n)$ with $\bar{D}_\nu \doteq \frac{\partial}{\partial x^\nu} + \alpha^\nu \frac{\partial}{\partial \theta}$ and $x^{n+1} = U^+_\alpha(x,\theta)$ one has $\Psi_\alpha(\overline{x}) = \bar{D} U^+_\alpha(x,\theta)$. Setting $U \doteq U^+_\alpha$ and $U_\theta(x,\theta) \doteq \lim_{\tau \downarrow 0} \frac{1}{\tau}(U(x, \theta + \tau) - U(x,\theta))$, the desired equality (5) translates to

$$\int_{\overline{E}^\circ_{0,e_i}} F_{p^i}(x, U, \bar{D}U) U_\theta\, dx^1 \overset{i}{\dots} dx^n d\theta = \int_{\overline{E}_0} F_{p^i}(x, U, \bar{D}U) U_\theta\, dx d\theta\ , \tag{6}$$

where $\overline{E}^\circ_{0,e_i} \doteq \overline{E}_0 \cap \langle e_i \rangle$. This last equality itself is obtained by substituting the Euler equation $F_u(x, U, \bar{D}U) = \sum \bar{D}_\nu F_{p^\nu}(x, U, \bar{D}U)$ in

$$0 = \int_{\overline{E}_0} x^i \frac{\partial}{\partial \theta} F(x, U, \bar{D}U) dx d\theta = \int_{\overline{E}_0} x^i (F_u \cdot U_\theta + \sum_{\nu=1}^{n} F_{p^\nu} \cdot \bar{D}_\nu U_\theta) dx d\theta$$

and integrating the term $x^i U_\theta \sum \bar{D}_\nu F_{p^\nu}$ partially n times with respect to $\bar{D}_\nu$. Using the periodicities of U and F all summands up to the one with $i = \nu$ cancel and left- and right-hand side of (6) remain with opposite sign. This establishes formula (5) and by the main theorem the corollary is proved. □

Moser presented in [15, 16] a regularized variational problem for minimal foliations of T^{n+1}. The corresponding regularized minimal foliations tend uniformly on compact sets to our generalized minimal foliation with a parameter $\varepsilon \to 0$ [21]. Since the corresponding functions Ψ_α^ε describing the regularized foliation is differentiable for $\varepsilon > 0$, the partial derivatives of the minimal average action $A^\varepsilon(\alpha)$ with respect to the regularized variational problem may be computed explicitly by

$$\frac{\partial}{\partial \alpha^i} A^\varepsilon(\alpha) = \int_{\overline{E}_0} F^\varepsilon_{p^i}(\overline{x}, \Psi^\varepsilon_\alpha)\, d\overline{x}\ , \qquad 1 \le i \le n\ ,$$

see [16, (8.10)]. According to the corollary, the same formula remains true in the degenerate case $\varepsilon = 0$, although $\Psi_\alpha = \lim_{\varepsilon\to 0} \Psi_\alpha^\varepsilon$ need not be differentiable anymore. In [21] it was shown that for $\varepsilon \to 0$ the minimal average action $A^\varepsilon(\alpha)$ of the regularized problem tends pointwise to $A(\alpha)$. If $\mathscr{M}_\alpha$ gives rise to a foliation, according to the corollary, the derivative $DA^\varepsilon(\alpha)$ likewise converges to $DA(\alpha)$. Using Theorem 2 and the convexity of A one even shows more :

$DA^\varepsilon(\alpha)$ tends pointwise to $DA(\alpha)$ at $\alpha \in \mathbf{R}^n$ with $\varepsilon \to 0$ if and only if $\mathscr{M}_\alpha$ is totally ordered.

Finally we replace the sum in formula (2) by a sum which surely is absolutely convergent. The idea is to integrate not over the domain E^+_{0,e_i} cut out of some natural periodicity domain $E_{\alpha,\beta}$ of the solutions in $\mathscr{M}_{\alpha,\beta}$ but to integrate over such a periodicity domain itself.

Let $\mathscr{B}_\circ \doteq \{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbf{R}^n$. We still assume the standard situation where $V_\alpha = \mathbf{R}^r \times \mathbf{R}^{n-r}$ with $r = \dim_{\mathbf{R}} V_\alpha$. By $E_\alpha \subseteq \mathbf{R}^n$ we denote the fundamental domain of $\mathbf{R}^n/\Gamma_\alpha$ spanned by the standard basis $\mathscr{B}_\circ$ on V_α, i.e.

$$E_\alpha = \ \mathrm{span}_{[0,1]}\{s^1 e_1, \ldots, s^r e_r\} \times \ \mathrm{span}_{\mathbf{R}}\{e_{r+1}, \ldots, e_n\} \ \text{ if } r \ge 1 \tag{7}$$

and $E_\alpha = \mathbf{R}^n$ if $r = 0$. Notice that for $\beta \in \mathscr{B}_\circ \cap V_\alpha$ the set $E^+_{\alpha,\beta}$ is a genuine supset of E_α while for $\beta \in \mathscr{B}_\circ \cap V_\alpha^\perp$ the set is 'half' of E_α. For $\alpha = 0$ one has in particular $E_0 = [0,1]^n$.

If $\beta \in \mathscr{B}_\circ$ is an arbitrary standard unit vector we consider the sublattice $\Gamma_{\alpha,\beta} = \Gamma_\alpha \cap \langle\beta\rangle^\perp$ with the corresponding fundamental domain $E_{\alpha,\beta} \doteq E_\alpha + \mathbf{R}\beta$ of the quotient space $\mathbf{R}^n/\Gamma_{\alpha,\beta}$. Within $E_{\alpha,\beta}$ the following subsets are specified:

$$E^\circ_{\alpha,\beta} \doteq E_\alpha \cap \langle\beta\rangle^\perp \ \text{ and } \ E^\pm_{\alpha,\beta} \doteq E^\circ_{\alpha,\beta} \pm \mathbf{R}^+\beta\ .$$

In particular $E^-_{\alpha,\beta} = E^+_{\alpha,-\beta}$. In case $\alpha = 0$ and $\beta \in \mathscr{B}_\circ$ we have $E^\circ_{0,\beta} = [0,1]^n \cap \langle\beta\rangle^\perp$ and $|E^\circ_{0,\beta}| = 1$. Moreover $\Gamma_{0,\beta} = \{k \in \mathbf{Z}^n : k\beta = 0\}$. On the set of gaps $\Gamma_{\alpha,\beta}$ we define the equivalence relation

$$G_1 \overset{\Gamma_{0,\beta}}{\sim} G_2 \ :\Longleftrightarrow\ \exists\, \overline{k} \in \Gamma_{0,\beta} \times \mathbf{Z} \ \text{ such that } \ G_1 = G_2 - \overline{k}\ ,$$

where G_1, $G_2 \in \mathscr{G}_{\alpha,\beta}$. Let $\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}$ be a system of representatives of $\overset{\Gamma_{0,\beta}}{\sim}$. Now, for every $G \in \mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}$ one has $B_G(E^+_{\alpha,\beta}) = \sum B_H(E^+_{0,\beta})$, where the sum has to be taken over all $H \in \mathscr{G}_{\alpha,\beta}$ with $H \overset{\Gamma_{0,\beta}}{\sim} G$. Formula (2) therefore rewrites with $\beta = e_i$ in the form

$$D_\beta A(\alpha) = \int_{\mathscr{F}_{\alpha,\beta}(E^\circ_{0,\beta})} \beta F_p(\overline{x}, \Psi_{\alpha,\beta})\, dx^1 \overset{\beta}{\wedge\ldots} dx^{n+1} + \sum_{G \in \mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} B_G(E^+_{\alpha,\beta})\,, \tag{8}$$

where the sum may be shown to converge absolutely.

4 Proof of the qualitative behavior

We deduce Theorem 2 and 3 from the main theorem. In order to compare left- and right-sided derivative $D_\beta A(\alpha)$ and $D_{-\beta}A(\alpha)$, respectively, we expand the domain of integration $E^+_{\alpha,\pm\beta}$ to the common periodicity domain $E_{\alpha,\beta} = E^+_{\alpha,\beta} \cup E^+_{\alpha,-\beta}$ of solutions in $\mathscr{M}_{\alpha,\beta}$ and $\mathscr{M}_{\alpha,-\beta}$. The comparison of $\int_{E_{\alpha,\beta}} F(x,u,Du)dx$ with $u \in \mathscr{M}_{\alpha,\pm\beta}$ together with a standard reasoning counting the order of growth of the variational integrals on $E_{\alpha,\beta}$ will prove the first part of Theorem 2. The strict inequality in the second part of Theorem 2 is based on the maximum principle for partial elliptic differential equations in addition. The proof of Theorem 3 uses a further estimate of the size of the gaps defined by $\mathscr{M}(\overline{a}_1)$.

Choose $\alpha \in \mathbf{R}^n$. Without loss of generality we assume that the standard basis $\mathscr{B}_\circ$ of $\mathbf{R}^n$ is α-admissible for the chosen α. Reordering the basis this is equivalent to the standard situation where $V_\alpha = \mathbf{R}^r \times \{0\}^{n-r}$ with $r = \dim V_\alpha$. We discuss briefly the question of existence of $B_G(E^+_{\alpha,\beta})$ as well as of the sum in (9).

First, the integrand in $B_G(E^+_{\alpha,\beta})$ may be estimated uniformly by

$$\begin{aligned}&|F(x,u^-_G,Du^-_G) - F(x,u^+_G,Du^+_G)| \le \\ &\qquad \le \text{const} \cdot |u^+_G(x) - u^-_G(x)| \cdot (\max|F_u(\overline{x},p)| + \max\|F_p(\overline{x},p)\| + 1)\,.\end{aligned}$$

To deduce this inequality apply the fundamental theorem of calculus and the estimate of Lemma 1 (compare formula (15), sect. 4.2). The existence of $B_G(E^+_{\alpha,\beta})$ now follows from $\mathrm{vol}_{n+1}\{\overline{x} \in G : x \in E^+_{\alpha,\beta}\} \le \mathrm{vol}_{n+1} T^{n+1} = 1$.

Second, we give a rough idea why $\sum_{G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} B_G(E^+_{\alpha,\beta})$ is absolutely convergent and thus independent of the order of summation. Since each gap $G \in \mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}$ projects injectively into $\mathbf{R}^n/\Gamma_{0,\beta}$ the union of all $\{\overline{x} \in G : x \in E^\circ_{\alpha,\beta}\}$ with $G \in \mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}$ has finite n-dimensional volume. This would no longer be true in general if $E^\circ_{\alpha,\beta}$ were replaced by $E^+_{\alpha,\beta}$. However, in order to prove the finiteness of $\sum_{G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} |B_G(E^+_{\alpha,\beta})|$ it is by minimality of $u^\pm_G$ enough to argue with the boundary $E^\circ_{\alpha,\beta}$ of $E^+_{\alpha,\beta}$ (mod $\mathbf{Z}^n$) instead of $E^+_{\alpha,\beta}$ itself. For, the standard reasoning using the minimality as well as the asymptoticity and periodicity of u^-_G and u^+_G on $E^\circ_{\alpha,\beta}$ (see the proof of Theorem 2 in the next section) leads to an estimate of the form $|B_G(E^+_{\alpha,\beta})| \le \text{const} \cdot \mathrm{vol}_{n+1}\{\overline{x} \in G : x \in E^\circ_{\alpha,\beta}\}$.

4.1 Proof of the (non-)differentiability in a particular direction

The differentiability criterion of Theorem 2 and 3 will be deduced from the explicit formula stated in the main theorem. We say that some orthonormal basis $\mathscr{B} \subset S_Q^{n-1} = \{\frac{x}{\|x\|} \in S^{n-1} : x \in \mathbf{Q}^n \setminus \{0\}\}$ of $\mathbf{R}^n$ is **admissible** for $\alpha \in \mathbf{R}^n$ (short: α-admissible) if $\mathscr{B} \subseteq V_\alpha \cup V_\alpha^\perp$. Recall that $V_\alpha = \mathrm{span}_{\mathbf{R}} \Gamma_\alpha = \mathrm{span}_{\mathbf{R}}\{k \in \mathbf{Z}^n : \alpha k \in \mathbf{Z}\} \subseteq \mathbf{R}^n$. We assume the standard situation where the standard basis $\mathscr{B}_\circ$ is admissible for $\alpha \in \mathbf{R}^n$. i.e. $\mathscr{B}_\circ \subseteq V_\alpha \cup V_\alpha^\perp$. If $\beta \in \mathscr{B}_\circ$ is a standard unit vector we abbreviate

$$B(\alpha,\beta) \doteq \int_{\mathscr{T}_{\alpha,\beta}(E^\circ_{0,\beta})} \beta F_p(\overline{x}, \Psi_{\alpha,\beta})\, d\overline{x} + \sum_{G \in \mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} B_G(E^+_{\alpha,\beta}) \,. \tag{9}$$

corresponding to the right-hand side of (8). By the main theorem $D_\beta A(\alpha) = B(\alpha,\beta)$.

For a convex function on $\mathbf{R}^n$, the set of directions $\beta \in \mathbf{R}^n$ in which left- and right-hand sided derivative at some point $\alpha \in \mathbf{R}^n$ coincide, is a linear subspace. In order to deduce Theorem 2 from the main theorem it is therefore enough to show

$$B(\alpha,\beta) + B(\alpha,-\beta) = 0 \;\text{ if } \beta \in \mathscr{B}_\circ \cap V_\alpha^\perp \;\text{ and }\; > 0 \;\text{ if } \beta \in \mathscr{B}_\circ \cap V_\alpha \,.$$

Note that by the assumption on α one has $\mathrm{span}_{\mathbf{R}}(\mathscr{B}_\circ \cap V_\alpha^\perp) = V_\alpha^\perp$.

Proof of Theorem 2 for $\beta \in \mathscr{B}_\circ \cap V_\alpha^\perp$. Since in this case $\mathscr{M}_{\alpha,\beta} = \mathscr{M}_{\alpha,-\beta} = \mathscr{M}(\overline{a}_1)$ and $E_{\alpha,\beta} = E_\alpha$ one obtains

$$B(\alpha,\beta) + B(\alpha,-\beta) = \sum_{G \in \mathscr{G}_\alpha/\Gamma_{0,\beta}} (B_G(E^+_{\alpha,\beta}) + B_G(E^-_{\alpha,\beta})) = \sum_{G \in \mathscr{G}_\alpha/\Gamma_{0,\beta}} B_G(E_\alpha) \,,$$

where $\mathscr{G}_\alpha \doteq \mathscr{G}_{\alpha,\beta} = \mathscr{G}_{\alpha,-\beta}$ is the set of gaps of $\mathscr{M}(\overline{a}_1)$ lying between $u_\alpha - 1$ and u_α, u_α some fixed solution in $\mathscr{M}(\overline{a}_1)$.

We show that for each $G \in \mathscr{G}_\alpha$ one has $B_G(E_\alpha) = 0$:
Suppose $B_G(E_\alpha) = \int_{E_\alpha} (Fx, u_G^-, Du_G^-) - F(x, u_G^+, Du_G^+))\, dx \geq 2\varepsilon$ for some $\varepsilon > 0$. For $Q \doteq [-1,1]^n$ and $\tau_\circ$ large enough the restriction to the compact domain $E_\alpha \cap \tau_\circ Q$ still satisfies $B_G(E_\alpha \cap \tau_\circ Q) \geq \varepsilon$. By the periodicity of G in the directions $e_1, \ldots, e_r$ one concludes $B_G(\tau Q) \geq (\varepsilon \frac{\tau}{\tau_\circ})^r$ for $\tau \geq \tau_\circ$ large enough. This says that the variational integral of u_G^+ over τQ is of order τ^r less than the variational integral of u_G^- over the same domain τQ.

Now, a contradiction to the minimality of u_G^- may be deduced. For this purpose we construct compact variations of u_G^- supported by $(\tau+1)Q$ and agreeing with u_G^+ on τQ. The additional portion of these variations joining u_G^- with u_G^+ on the set $(\tau+1)Q \setminus \tau Q$ may be estimated from above by a factor const $\cdot\, \tau^{r-1}$ as follows:

Since the set $\{(x, x^{n+1}) \in G : x \in E_\alpha\}$ projects injectively into T^{n+1} its volume is ≤ 1. The functions u_G^- and u_G^+ therefore have to converge in the directions $e_{r+1}, \ldots, e_n$ to each other in a way insuring $\int_{\mathrm{bd}(\tau Q)} |u_G^+ - u_G^-|\, dx \leq$ const $\cdot\, \tau^{r-1}$ for infinitely many $\tau \in \mathbf{N}$. By $\mathrm{bd}(\tau Q)$ we denote the boundary of τQ. Recall that $G = (u_G^-, u_G^+)$ is periodic in the directions $e_1, \ldots, e_r$.

While the win for the variational integral of u_G^- is of order τ^r if it would agree with u_G^+ on τQ, the loss of joining $u_G^+|_{\tau Q}$ to a compact variation of u_G^- is only of order τ^{r-1}. Obviously, this is a contradiction to the minimality of u_G^- for τ large enough and proves $B_G(E_\alpha) \leq 0$.

The technique of counting orders presented here will be cited as the **standard reasoning**. For details see [4], in particular (6.9)-(6.13). Interchanging the rôles of u_G^- and u_G^+ yields $B_G(E_\alpha) = 0$.

Under the assumption $\mathscr{B}_\circ = \{e_1, \ldots, e_n\}$ being α-admissible, we therefore have shown

$$B(\alpha, \beta) + B(\alpha, -\beta) = 0 \;\text{ for all } \beta \in \mathscr{B}_\circ \cap V_\alpha^\perp\,. \quad \square \tag{10}$$

If in case $\beta \in \mathscr{B}_\circ \cap V_\alpha$ we add $B(\alpha, \beta)$ and $B(\alpha, -\beta)$, the integral term in (9) will no longer cancel. However, we may transform $B(\alpha, \beta)$ such that it again has this property. This allows us to establish the positivity of $B(\alpha, \beta) + B(\alpha, -\beta)$.

To $\beta \in \mathscr{B}_\circ \cap V_\alpha$ we associate $\overline{a}_2 \in S^n \cap \overline{V}_\alpha$ according to sect. 2.2. A subset $\mathscr{P} \subseteq \mathscr{M}(\overline{a}_1, \overline{a}_2)$ is called a **filtration** (of $\mathscr{M}(\overline{a}_1, \overline{a}_2)$) if it is invariant under the $\mathbf{Z}^{n+1}$-action T on $\mathscr{M}(\overline{a}_1, \overline{a}_2)$ and if to every gap $G \in \mathscr{G}_\alpha$ there exists $u \in \mathscr{P}$ with graph $u \subseteq G$. By $\mathscr{G}_{\mathscr{P}}$ we denote the set of gaps of

$$\bigcup \{ \text{ graph } u : u \in \mathscr{M}(\overline{a}_1) \cup \mathscr{P} \} \subseteq \mathbf{R}^{n+1}$$

lying within $(u_\alpha - 1, u_\alpha) = \{(x, x^{n+1}) \in \mathbf{R}^{n+1} : u_\alpha(x) - 1 < x^{n+1} < u_\alpha(x), x \in \mathbf{R}^n\}$. A filtration $\mathscr{P} \subseteq \mathscr{M}(\overline{a}_1, \overline{a}_2)$ is called **discrete** if $\mathscr{P}/\Gamma_{0,\beta}$ is discrete with respect to the C^1-topology. Let Ψ_α be the restriction of $\Psi_{\alpha,\beta}$ to $\mathscr{F}_\alpha(\mathbf{R}^n)$, i.e. $\Psi_\alpha(\overline{x}) = Du(x)$, where $u \in \mathscr{M}(\overline{a}_1)$ with $u(x) = x^{n+1}$.

Lemma 2 *Choose $\alpha \in \mathbf{R}^n$ such that the standard basis $\mathscr{B}_\circ$ is α-admissible, i.e. $\mathscr{B}_\circ \subseteq V_\alpha \cup V_\alpha^\perp$. Choose $\beta \in \mathscr{B}_\circ \cap V_\alpha$. If the filtrations $\mathscr{P}_\pm \subseteq \mathscr{M}(\overline{a}_1, \pm\overline{a}_2)$ are discrete one has*

$$B(\alpha, \pm\beta) = \pm \int_{\mathscr{F}_\alpha(E_{0,\beta}^\circ)} \beta F_p(\overline{x}, \Psi_\alpha)\, d\overline{x} + \sum_{H \in \mathscr{G}_{\mathscr{P}_\pm}/\Gamma_{0,\beta}} B_H(E_{\alpha,\beta}^\pm)\,.$$

The proof is postponed to sect. 4.2.

The summation term in the lemma may be simplified by interpreting the partial sum

$$\sum_{\substack{H \in \mathscr{G}_{\mathscr{P}_\pm}/\Gamma_{0,\beta} \\ H \subset G}} B_H(E_{\alpha,\beta}^\pm)\,, \quad G \in \mathscr{G}_\alpha/\Gamma_{0,\beta}\,. \tag{11}$$

By discreteness of $\mathscr{P}_\pm$ this partial sum is an infinite sum of telescope and therefore reduces to one integral.

For any $G = (u_G^-, u_G^+) \in \mathscr{G}_\alpha/\Gamma_{0,\beta}$ fix $u_{G,\pm} \in \mathscr{P}_\pm$ with graph $u_{G,\pm} \subseteq G$. Let $s \in \mathbf{N}$ be the smallest natural number such that $s\beta \in \Gamma_\alpha$. Equivalently, s is characterized by $\alpha\beta = \frac{r}{s}$ where $r \in \mathbf{Z}$ and $s \in \mathbf{N}$ are relatively prime. If $\alpha\beta = 0$ we have $s = 1$.

Note that $u_G^+ \in \mathscr{M}(\overline{a}_1)$ is periodic mod 1 with respect to translations $\tau s\beta$, $\tau \in \mathbf{Z}$, and that $\bigcup_{\tau\in\mathbf{Z}} \tau s\beta + E_\alpha = E_{\alpha,\beta}$. Since $u_{G,+}$ and $u_{G,-}$ are asymptotic in the direction β and $-\beta$ respectively for any $G \in \mathscr{G}_\alpha$, the expression (11) simplifies to

$$B_G^{\pm}(E_{\alpha,\beta}) \doteq \lim_{\tau\in\mathbf{N}} \int_{E^\circ_{\alpha,\beta}+[-\tau,\tau]s\beta} (F(x, u_{G,\pm}, Du_{G,\pm}) - F(x, u_G^+, Du_G^+))\, dx\, .$$

The limit exists since it corresponds to a special ordering of summation in (11) and since this last sum is absolutely convergent. From the lemma we conclude

Corollary 3 *Let $\mathscr{B}_\circ$ be α-admissible for $\alpha \in \mathbf{R}^n$. For any $\beta \in \mathscr{B}_\circ \cap V_\alpha$ one has*

$$B(\alpha, \pm\beta) = \pm \int_{\mathscr{T}_\alpha(E^\circ_{0,\beta})} \beta F_p(\overline{x}, \Psi_\alpha)\, d\overline{x} + \sum_{G\in\mathscr{G}_\alpha/\Gamma_{0,\beta}} B_G^{\pm}(E_{\alpha,\beta})\, ,$$

$$B(\alpha, \beta) + B(\alpha, -\beta) = \sum_{G\in\mathscr{G}_\alpha/\Gamma_{0,\beta}} B_G^+(E_{\alpha,\beta}) + B_G^-(E_{\alpha,\beta})\, .$$

Remark Since by the main theorem $D_\beta A(\alpha) = B(\alpha, \beta)$ in case that the standard basis $\mathscr{B}_\circ$ is α-admissible, formula (2) takes the new form

$$D_{e_i} A(\alpha) = \int_{\overline{E}_0} F_{p^i}(x, U, \bar{D}U) U_\theta\, d\overline{x} d\theta + \sum_{G\in\mathscr{G}_\alpha/\Gamma_{0,e_i}} B_G^{\pm}(E_{\alpha,e_i})$$

We used (5) and the notations there. In case that $\mathscr{M}_\alpha$ does not have gaps and thus $\mathscr{G}_\alpha/\Gamma_{0,e_i} = \emptyset$, the formula above is the same as the formula for $\frac{\partial}{\partial\alpha^i} M^\varepsilon(\alpha)$ in [16, ch. 8]. □

Proof of Theorem 2 for $\beta \in \mathscr{B}_\circ \cap V_\alpha$. By the main theorem it is enough to show that

$$B_G^+(E_{\alpha,\beta}) + B_G^-(E_{\alpha,\beta}) > 0 \tag{12}$$

holds for each $G \in \mathscr{G}_\alpha$. If $G \in \mathscr{G}_\alpha$ define $G+ \doteq (\max(u_{G,-}, u_{G,+}),\, u_G^+)$ and $G- \doteq (u_G^-,\, \min(u_{G,-}, u_{G,+}))$. Using the fact that for any $\tau \in \mathbf{Z}$

$$\int_{E^\circ_{\alpha,\beta}+[\tau-1,\tau]s\beta} F(x, u_G^+, Du_G^+)\, dx = \int_{E^\circ_{\alpha,\beta}+[\tau-1,\tau]s\beta} F(x, u_G^-, Du_G^-)\, dx$$

$(= |E_\alpha| \cdot A(\alpha))$, we obtain

$$B_G^+(E_{\alpha,\beta}) + B_G^-(E_{\alpha,\beta}) = B_{G+}(E_{\alpha,\beta}) - B_{G-}(E_{\alpha,\beta})\, .$$

By the standard technique explained in the preceding part using the minimality of u_G^+ and u_G^- one proves $\pm B_{G\pm}(E_{\alpha,\beta}) \geq 0$. However, since u_{G-} and u_{G+} intersect, the maximum principle states that neither $\max(u_{G-}, u_{G+})$ nor $\min(u_{G-}, u_{G+})$ can be minimal on $E_{\alpha,\beta}$, see e.g. [4, (6.1)]. One therefore actually has $\pm B_{G\pm}(E_{\alpha,\beta}) > 0$, from which (12) follows. □

Next we prove Theorem 3. By the main theorem $(D_\beta + D_{-\beta})A(\alpha) = B(\alpha,\beta) + B(\alpha,-\beta)$ for any $\beta \in \mathscr{B}_\circ$ and according to Corollary 3 one has to estimate the summands $B_G^+(E_{\alpha,\beta}) + B_G^-(E_{\alpha,\beta})$.

If $t \in \mathbf{R}$ and $G \in \mathscr{G}_\alpha$ we denote by ω_G^t the n-dimensional volume of $\{\overline{x} \in G : x \in E_{\alpha,\beta}\, , \, x\beta = t\}$.

Lemma 3 *Let $K \subseteq \mathbf{R}^n$ be compact. Suppose $\mathscr{B}_\circ$ is α-admissible for $\alpha \in K$ and $\beta \in \mathscr{B}_\circ \cap V_\alpha$. There is a constant depending on F and K only such that for all $G \in \mathscr{G}_\alpha$ and all $t \in \mathbf{R}$*

$$0 \leq B_G^+(E_{\alpha,\beta}) + B_G^-(E_{\alpha,\beta}) \leq \; const \; (\omega_G^t)^2 \; .$$

Proof of Theorem 3 The assumption $\alpha^i = \alpha\beta \in \mathbf{Q}$ with $\beta = e_i \in \mathscr{B}_\circ$ in Theorem 3 is equivalent to $\beta \in \mathscr{B}_\circ \cap V_\alpha$. According to Corollary 3 and Lemma 3

$$(D_\beta + D_{-\beta})A(\alpha) \leq \text{ const } \cdot \sum_{G \in \mathscr{G}_\alpha / \Gamma_{0,\beta}} (\omega_G^{t_G})^2 \tag{13}$$

holds with any $t_G \in \mathbf{R}$.

Let $\alpha\beta = \frac{r}{s} \in \mathbf{Q}$ be as below of Lemma 2. Remember that G is periodic mod 1 under translations with $s\beta \in \Gamma_\alpha \subseteq \mathbf{Z}^n$.

To every $G \in \mathscr{G}_\alpha$ let $0 \leq t_G < s$ be such that $\omega_G^{t_G} \leq \omega_G^t$ for all $0 \leq t < s$ and thus by periodicity for all $t \in \mathbf{R}$. One estimates

$$\sum_{G \in \mathscr{G}_\alpha / \Gamma_{0,\beta}} \omega_G^{t_G} \leq \sum_{G \in \mathscr{G}_\alpha / \Gamma_{0,\beta}} \omega_G^\circ \leq 1 \, , \tag{14}$$

where the second inequality holds because $\bigcup_{G \in \mathscr{G}_\alpha / \Gamma_{0,\beta}} \{\overline{x} \in G : x \in E_{\alpha,\beta}^\circ\}$ projects injectively by $\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}/\mathbf{Z}^{n+1}$ into the face $\overline{E}_{0,\beta}^\circ$ of $\overline{E}_0 = [0,1]^{n+1}$. Since the set $\{\overline{x} \in G : x \in E_\alpha\}$ projects injectively into $\overline{E}_0$ one has $\int_0^s \omega_G^t \, dt \leq 1$ for any $G \in \mathscr{G}_\alpha$. The special choice of t_G implies $\omega_G^{t_G} \leq \frac{1}{s}$. Replacing one factor $\omega_G^{t_G}$ in (13) by $\frac{1}{s}$ and applying (14) we get

$$(D_\beta + D_{-\beta})A(\alpha) \leq \text{ const } \frac{1}{s} \, ,$$

proving Theorem 3. □

4.2 Appendix: a chapter of virtuous calculus

Proof of Lemma 2 We restrict to the case $+\beta$. Note that the lemma is trivial if $\mathscr{M}(\overline{a}_1,\overline{a}_2)/\Gamma_{0,\beta}$ itself is discrete with respect to the C^1-topology. To prove the general case one has to show that

$$\sum_{H\in\mathscr{G}_{\mathscr{P}_+}/\Gamma_{0,\beta}} B_H(E^+_{\alpha,\beta}) = \int_{\mathscr{F}_{\alpha,\beta}\setminus\mathscr{F}_\alpha(E^\circ_{0,\beta})} \beta F_p(\overline{x},\Psi_{\alpha,\beta})\,d\overline{x} + \sum_{G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} B_G(E^+_{\alpha,\beta}) ,$$

where $\mathscr{F}_{\alpha,\beta}\setminus\mathscr{F}_\alpha(\Omega) \doteq \mathscr{F}_{\alpha,\beta}(\Omega)\setminus\mathscr{F}_\alpha(\Omega)$ for $\Omega\subseteq\mathbf{R}^n$.

The technique is to refine the fixed discrete filtration $\mathscr{P}_+$ successively until it will fill up the set $\mathscr{M}(\overline{a}_1,\overline{a}_2)$. Of course, if $\mathscr{P}\supseteq\mathscr{P}_+$ and $\mathscr{P}$ is a discrete filtration, the left-hand side above will not change. It is therefore enough to show that for such successive refinements of $\mathscr{P}_+$ the right-hand side will be approximated by the right-hand side. The integral term in the formula may be interpreted as an infinitesimal version of $\sum B_G(E^+_{\alpha,\beta})$ when the volume of the gaps G tends to 0.

For any discrete filtration $\mathscr{P}\subseteq\mathscr{M}(\overline{a}_1,\overline{a}_2)$ we decompose $\mathscr{G}_\mathscr{P}$ into complementary sets $\mathscr{G}_\mathscr{P} = \mathscr{G}^\circ_\mathscr{P} \,\dot{\cup}\, \mathscr{G}^1_\mathscr{P}$ with $\mathscr{G}^\circ_\mathscr{P} \doteq \mathscr{G}_{\alpha,\beta}\cap\mathscr{G}_\mathscr{P}$ and $\mathscr{G}^1_\mathscr{P} \doteq \mathscr{G}_\mathscr{P}\setminus\mathscr{G}^\circ_\mathscr{P}$. A sequence of discrete filtrations $\mathscr{P}\subseteq\mathscr{M}(\overline{a}_1,\overline{a}_2)$ is said to converge to $\mathscr{M}(\overline{a}_1,\overline{a}_2)$, $\mathscr{P}\to\mathscr{M}(\overline{a}_1,\overline{a}_2)$, if $\bigcup\{G\in\mathscr{G}^\circ_\mathscr{P}\}\to\bigcup\{G\in\mathscr{G}_{\alpha,\beta}\}$ with respect to the Hausdorff metric on subsets of $\mathbf{R}^{n+1}$. In a complementary formulation, using the regularity of the minimal solutions, $\mathscr{P}\to\mathscr{M}(\overline{a}_1,\overline{a}_2)$ iff

$$\left|\sum_{G\in\mathscr{G}^1_\mathscr{P}} (u^+_G - u^-_G)(x) - \operatorname{vol}_1(\mathscr{F}_{\alpha,\beta}\setminus\mathscr{F}_\alpha(x))\right| \longrightarrow 0$$

uniformly in $E^+_{\alpha,\beta}$.

We rewrite the expression $B_G(E^+_{\alpha,\beta})$ applying simple calculus. First

$$\begin{aligned} F(x,u^-_G,Du^-_G) &= F(x,u^+_G,Du^+_G) + \int_{u^+_G(x)}^{u^-_G(x)} F_u(x,\xi,Du^+_G)\,d\xi + \\ &+ \sum_{i=1}^n \frac{\partial}{\partial x^i}(u^-_G - u^+_G)F_{p^i}(x,u^-_G,Du^-_G) + O(\|Du^-_G - Du^+_G\|^2) \end{aligned} \tag{15}$$

with an error term bounded by

$$|O(\|Du^-_G - Du^+_G\|^2)| \le \delta^{-1}\frac{n(n+1)}{2}\|Du^-_G - Du^+_G\|^2 \le \text{const}\cdot|u^-_G - u^+_G|^2 .$$

$\delta>0$ is the constant in $(F2)$. According to Lemma 1, using the periodicity and the total ordering of $\mathscr{M}(\overline{a}_1,\overline{a}_2)$, the constant may be chosen independently of $x\in E^+_{\alpha,\beta}$ and $G\in\mathscr{G}_\mathscr{P}$. One obtains

$$B_G(E^+_{\alpha,\beta}) = \int_{E^+_{\alpha,\beta}} \int_{u^+_G(x)}^{u^-_G(x)} F_u(x,\xi,Du^+_G)\,d\xi dx + \tag{16}$$

$$+ \int_{E^+_{\alpha,\beta}} \sum_{i=1}^n \frac{\partial}{\partial x^i}(u^-_G - u^+_G)F_{p^i}(x,u^-_G,Du^-_G)\,dx + \int_{E^+_{\alpha,\beta}} O(|u^-_G - u^+_G|^2)dx\ .$$

Summing up the right-hand side over all $G \in \mathcal{G}_{\mathcal{P}}/\Gamma_{0,\beta}$ one has to cope with the difficulty that this sum need no longer to converge absolutely. We fix some discrete filtration $\mathcal{P}_+^{\circ} \subseteq \mathcal{M}(\overline{a}_1,\overline{a}_2)$. Since for each $H \in \mathcal{G}_{\mathcal{P}_+^{\circ}}$ the $(n+1)$-dimensional volume of $\{\overline{x} \in H : x \in E^+_{\alpha,\beta}\}$ is finite, we may first restrict the sum to all $G \subseteq H$ and then add over all $H \subseteq \mathcal{G}_{\mathcal{P}_+^{\circ}}/\Gamma_{0,\beta}$.

Let us consider a sequence of discrete filtrations $\mathcal{P}$ converging to $\mathcal{M}(\overline{a}_1,\overline{a}_2)$. Fix $H \in \mathcal{G}_{\mathcal{P}_+^{\circ}}$. In order to determine the limit $\lim_{\mathcal{P}} \sum B_G(E^+_{\alpha,\beta})$ we treat each of the three terms in (16) separately.

A) By elementary calculus one shows that in the limit $\mathcal{P} \to \mathcal{M}(\overline{a}_1,\overline{a}_2)$ we have

$$\lim_{\mathcal{P}} \sum_{\substack{G \in \mathcal{G}^1_{\mathcal{P}}/\Gamma_{0,\beta} \\ G \subseteq H}} \int_{E^+_{\alpha,\beta}} \int_{u^+_G(x)}^{u^-_G(x)} F_u(x,\xi,Du^+_G)\,d\xi dx =$$

$$= - \int_{H \cap \mathcal{F}_{\alpha,\beta} \setminus \mathcal{F}_\alpha(E^+_{\alpha,\beta})} F_u(\overline{x},\Psi_{\alpha,\beta})\,d\overline{x}\ .$$

To realize the limit process one uses the uniform estimates of Lemma 1 and the smoothness $(F1)$ of F.

B) In the limit $\mathcal{P} \to \mathcal{M}(\overline{a}_1,\overline{a}_2)$, after summation, the second term of (16) leads to

$$\lim_{\mathcal{P}} \sum_{\substack{G \in \mathcal{G}^1_{\mathcal{P}}/\Gamma_{0,\beta} \\ G \subseteq H}} \int_{E^+_{\alpha,\beta}} \sum_{i=1}^n \frac{\partial}{\partial x^i}(u^-_G - u^+_G)F_{p^i}(x,u^-_G,Du^-_G)\,dx =$$

$$= \int_{H \cap \mathcal{F}_{\alpha,\beta} \setminus \mathcal{F}_\alpha(E^\circ_{\alpha,\beta})} \beta F_p(\overline{x},\Psi_{\alpha,\beta})\,d\overline{x} + \int_{H \cap \mathcal{F}_{\alpha,\beta} \setminus \mathcal{F}_\alpha(E^+_{\alpha,\beta})} F_u(\overline{x},\Psi_{\alpha,\beta})\,d\overline{x}\ .$$

To check this equality one first has to integrate by part in all directions $e_i \in \mathcal{B}_\circ$ and to use that in the directions $\pm e_i$ the gap $H \in \mathcal{G}_{\alpha,\beta}$ either is periodic or converges to zero. Then apply the Euler equation

$$\sum_{i=1}^n \frac{\partial}{\partial x^i} F_{p^i}(x,u^-_G,Du^-_G) = F_u(x,u^-_G,Du^-_G)$$

and take the limit $\mathcal{P} \to \mathcal{M}(\overline{a}_1,\overline{a}_2)$.

C) Since for any $G \in \mathcal{G}^1_{\mathcal{P}}$ the function $|u^-_G - u^+_G|$ converges uniformly on $E^+_{\alpha,\beta}$ to 0 when $\mathcal{P} \to \mathcal{M}(\overline{a}_1,\overline{a}_2)$ and since the restriction of H to $E^+_{\alpha,\beta} \times \mathbf{R}$ has finite volume, one gets

$$\lim_{\mathscr{P}} \sum_{\substack{G\in\mathscr{G}^1_{\mathscr{P}}/\Gamma_{0,\beta}\\ G\subseteq H}} \int_{E^+_{\alpha,\beta}} O(|u^-_G - u^+_G|^2)\,dx = 0\,.$$

Adding the parts A) - C) it remains the term $\int \beta F_p(\overline{x}, \Psi_{\alpha,\beta}) d\overline{x}$ only. Summing over all $H \in \mathscr{G}_{\mathscr{P}_+^1}/\Gamma_{0,\beta}$ one obtains

$$\lim_{\mathscr{P}} \sum_{G\in\mathscr{G}^1_{\mathscr{P}}/\Gamma_{0,\beta}} B_G(E^+_{\alpha,\beta}) = \int_{\mathscr{F}_{\alpha,\beta}\setminus\mathscr{F}_\alpha(E^\circ_{0,\beta})} \beta F_p(\overline{x}, \Psi_{\alpha,\beta})\,d\overline{x}\,, \tag{17}$$

where the union $(\mathscr{F}_{\alpha,\beta} \setminus \mathscr{F}_\alpha(E^\circ_{\alpha,\beta}))/\Gamma_{0,\beta}$ of all integration domains is replaced by $\mathscr{F}_{\alpha,\beta} \setminus \mathscr{F}_\alpha(E^\circ_{0,\beta})$.

It remains to show that for the complementary set $\mathscr{G}^\circ_{\mathscr{P}}$ we have

$$\lim_{\mathscr{P}} \sum_{G\in\mathscr{G}^\circ_{\mathscr{P}}/\Gamma_{0,\beta}} B_G(E^+_{\alpha,\beta}) = \sum_{G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} B_G(E^+_{\alpha,\beta})\,.$$

The limit exists since $\sum_{G\in\mathscr{G}_{\mathscr{P}}/\Gamma_{0,\beta}} B_G(E^+_{\alpha,\beta})$ is independent of the discrete filtration $\mathscr{P}$ and since the complementary limit (17) exists. Since $\mathscr{G}^\circ_{\mathscr{P}} = \mathscr{G}_{\alpha,\beta} \cap \mathscr{G}_{\mathscr{P}} \subseteq \mathscr{G}_{\alpha,\beta}$ and $\mathscr{P} \to \mathscr{M}(\overline{a}_1, \overline{a}_2)$ the limit on the left-hand side defines a particular e order of summation of the sum at the right-hand side. □

Proof of Lemma 3 Choose $G = (u^-, u^+) \in \mathscr{G}_\alpha$ and assume $t = 0$. We define functions v^+ and v^- which coincide with u^- on $E^\circ_{\alpha,\beta}$ and which connect u^- with u^+ in the direction β respectively $-\beta$.

Let be $E^{+1}_{\alpha,\beta} \doteq \{x \in E_{\alpha,\beta} : 0 \le x\beta \le 1\}$, $E^{-1}_{\alpha,\beta} \doteq \{x \in E_{\alpha,\beta} : -1 \le x\beta \le 0\}$ and set

$$v^\pm(x) \doteq u^-(x) \pm x\beta(u^+ - u^-)(x) \quad \text{if } x \in E^{\pm 1}_{\alpha,\beta}\,.$$

Let us abbreviate

$$B^{\pm 1}_G \doteq \int_{E^{\pm 1}_{\alpha,\beta}} (F(x, v^\pm, Dv^\pm) - F(x, u^+, Du^+))\,dx\,.$$

We show with the standard reasoning using the minimality of $u^\pm_G$ that

$$B^{+1}_G + B^{-1}_G \ge B^+_G(E_{\alpha,\beta}) + B^-_G(E_{\alpha,\beta})\,.$$

In the following, the left-hand side will be estimated from above by $\text{const}\cdot(\omega^\circ_G)^2$, proving the lemma. Using elementary calculus one may substitute

$$F(x, v^\pm, Dv^\pm) = F(x, u^+, Du^+) + \int_{u^+(x)}^{v^\pm(x)} F_u(x, \xi, Du^+)\,d\xi + $$
$$+ \sum_{i=1}^n \frac{\partial}{\partial x^i}(v^\pm - u^+) F_{p^i}(x, v^\pm, Du^+) + O(\|Dv^\pm - Du^+\|^2)\,. \tag{18}$$

According to Lemma 1, the error term is bounded by

$$|O(\|Dv^{\pm} - Du^{+}\|^2)| \leq \delta^{-1}\frac{n(n+1)}{2}\|Dv^{\pm} - Du^{+}\|^2 \leq \text{ const} \cdot |u^{+} - u^{-}|^2 .$$

Moreover, since $F \in C^{2,\varepsilon}$, one has for $\xi \in [u^{-}(x), u^{+}(x)]$ the estimates

$$\begin{aligned} |F_u(x,\xi,Du^{+}) - F_u(x,u^{+},Du^{+})| &\leq \text{ const } |u^{+}(x) - u^{-}(x)| , \\ |F_{p^i}(x,v^{\pm},Du^{+}) - F_{p^i}(x,u^{+},Du^{+})| &\leq \text{ const } |u^{+}(x) - u^{-}(x)| . \end{aligned}$$

Inserting the last three estimates in (18), applying the mean value theorem and the fact $u^{-}(x) \leq v^{\pm}(x) \leq u^{+}(x)$, we obtain

$$B_G^{\pm 1} = \int_{E^{\pm 1}_{\alpha,\beta}} ((v^{\pm} - u^{+})F_u(x,u^{+},Du^{+}) +$$

$$+ \sum_{i=1}^{n} \frac{\partial}{\partial x^i}(v^{\pm} - u^{+})F_{p^i}(x,u^{+},Du^{+}) + O(|u^{+} - u^{-}|^2))\, dx .$$

Again, we integrate the summation term n times by part and substitute the Euler equation $\sum_{i=1}^{n} \frac{\partial}{\partial x^i} F_{p^i}(x,u^{+},Du^{+}) = F_u(x,u^{+},Du^{+})$. Because of the periodic respectively asymptotic behavior of u^{-} and u^{+} in the directions $e_i \in \mathscr{B}_\circ$ all boundary evaluation cancel up to one. It remains

$$B_G^{\pm 1} = \pm \int_{E^{\circ}_{\alpha,\beta}} (u^{+} - u^{-})\beta F_p(x,u^{+},Du^{+})\, dx + \int_{E^{\pm 1}_{\alpha,\beta}} O(|u^{+} - u^{-}|^2)\, dx. \qquad (19)$$

Using the second estimate of Lemma 1 we obtain

$$B_G^{+1} + B_G^{-1} \leq \text{ const } \int_{E^{-1}_{\alpha,\beta} \cup E^{+1}_{\alpha,\beta}} (u^{+} - u^{-})^2(x)\, dx \leq$$

$$\leq \text{ const } (\int_{E^{-1}_{\alpha,\beta} \cup E^{+1}_{\alpha,\beta}} (u^{+} - u^{-})(x)\, dx)^2 \leq \text{ const } (\omega_G^{\circ})^2 . \quad \square$$

5 Proof of the formula for the directional derivatives

We first prove the main theorem for all rational α (sect. 5.1). The general case is deduced by a limit process. We give a short overview of the proof.

If $\alpha \in \mathbf{Q}^n$ the problem of determining the derivative $D_\beta A(\alpha)$ essentially reduces to the one-dimensional case. This is due to the fact that by periodicity it is enough to consider the variational integral on a compact set times $\mathbf{R}\beta$. Evaluating explicitly the difference quotient with the help of calculus and taking the limit we show by some detour

$$D_\beta A(\alpha) = B(\alpha,\beta) , \quad \text{with } \alpha \in \mathbf{Q}^n , \beta \in (V_\alpha \cup V_\alpha^{\perp}) \cap S_Q^{n-1}$$

and $B(\alpha, \beta)$ in the sense of (9). This is the statement of the main theorem for rational α.

To prove the main theorem in the general case we have to establish this equality for arbitrary $\alpha \in \mathbf{R}^n$ and arbitrary $\beta \in (V_\alpha \cup V_\alpha^\perp) \cap S_Q^{n-1}$. The difficulty is that the directional derivative is not continuous in general. Let us fix some $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$ and $\beta \in (V_\alpha \cup V_\alpha^\perp) \cap S_Q^{n-1}$. The idea to overcome the difficulty is to look for a particular sequence $\alpha' \to \alpha$ with $\alpha' \in \mathbf{Q}^n$ such that

$$\lim_{\alpha' \to \alpha} D_\beta A(\alpha') = D_\beta A(\alpha) \qquad \text{as well as} \qquad \lim_{\alpha' \to \alpha} B(\alpha', \beta) = B(\alpha, \beta) . \tag{20}$$

To ensure the continuity of the limit $D_\beta(\alpha')$, $\alpha' \to \alpha$, it is by convexity of A enough to require

$$\lim_{\alpha' \to \alpha} \frac{\alpha' - \alpha}{\|\alpha' - \alpha\|} = \beta \qquad (\alpha' \in \mathbf{Q}^n) , \tag{21}$$

see Lemma 8. The continuity of the limit $B(\alpha', \beta)$, $\alpha' \to \alpha$, is more delicate. It is based on the continuity properties of the corresponding generalized foliations, i.e. whether $\mathscr{M}_{\alpha',\beta}$ tends to $\mathscr{M}_{\alpha,\beta}$ or not. To ensure this convergence one has to impose restrictions on the rational dependency of α': The approximating foliations e.g. have to exhibit at least the same periodicity as the foliation one wishes to approximate, i.e. $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$. If $\beta \in V_\alpha$ this condition will be weakened since it is no longer compatible with (21). This will be done in sect. 5.2.

It is easy to find a sequence $\alpha' \to \alpha$ such that the corresponding foliations converge and such that (21) and thus the first equality in (20) is satisfied. Fixing such a sequence we in a first step show

$$\lim_{\alpha' \to \alpha} B(\alpha', \beta) \le B(\alpha, \beta) ,$$

see Lemma 9. By our technique, however, the reversed inequality may not be ensured in a direct way. Now we use the first equality in (20) and the fact that we already proved $D_\beta A(\alpha') = B(\alpha', \beta)$ for $\alpha' \in \mathbf{Q}^n$. From the inequality above we deduce

$$D_\beta A(\alpha) \le B(\alpha, \beta) , \qquad \text{for } \beta \in (V_\alpha \cup V_\alpha^\perp) \cap S_Q^{n-1} , \alpha \in \mathbf{R}^n . \tag{22}$$

In a second step, if with the same α and β a sequence $\alpha'' \to \alpha$ satisfies $\Gamma_\alpha \subseteq \Gamma_{\alpha''}$ with $\alpha'' \in \mathbf{Q}^n$ (but in general violates (21)) we show

$$\lim_{\alpha'' \to \alpha} B(\alpha'', \beta) + B(\alpha'', -\beta) \ge B(\alpha, \beta) + B(\alpha, -\beta) ,$$

see Lemma 10. Since $D_\beta A(\alpha'') = B(\alpha'', \beta)$ for rational α'' and since by convexity of A the derivative $D_\beta A(\alpha)$ is upper semi-continuous in α we deduce

$$D_\beta A(\alpha) + D_{-\beta} A(\alpha) \ge B(\alpha, \beta) + B(\alpha, -\beta) .$$

Both steps together show that actually equality holds in (22). This establishes the main theorem in the general case.

5.1 The formula for the rational case

Given $\alpha \in \mathbf{Q}^n$, we determine the (one-sided) derivative $D_\beta A(\alpha)$ in the direction $\beta \in \mathscr{B}_\circ = \{e_1, \dots, e_n\}$. Let $\alpha \in \mathbf{Q}^n$ have components $\alpha^i = \frac{r^i}{s^i}$ with $r^i \in \mathbf{Z}$ and $s^i \in \mathbf{N}$ relatively prime. Assume $\beta = e_1$. For $\sigma \in \mathbf{Z}$ set

$$E^\sigma \doteq \mathrm{span}_{[0,1]}\{\sigma s^1 e_1, s^2 e_2, \dots, s^n e_n\} .$$

Note that $E^\circ = E^\circ_{\alpha,\beta}$, $E^1 = E_\alpha$ and $\bigcup_{\sigma\in\mathbf{Z}} E^\sigma = E_{\alpha,\beta}$. For $\sigma \in \mathbf{N}$ choose solutions $v^\sigma \in \mathscr{M}^{per}_{\alpha+\frac{\beta}{s\sigma}}$, $s \doteq s^1$, and fix $u_\alpha \in \mathscr{M}^{per}_\alpha$. If $\Omega \subseteq \mathbf{R}^n$ and $w \in W^{1,2}_{loc}(\mathbf{R}^n)$ we abbreviate

$$I(w,\Omega) \doteq \int_\Omega F(x,w,Dw)\,dx .$$

Since every solution v^σ is maximally periodic, E^σ is a periodicity domain for v^σ. In order to get the minimal average action it is enough to average the corresponding variational integral over one periodicity domain[20, 3.5]:

$$A(\alpha) = \frac{1}{|E^\sigma|} I(u_\alpha, E^\sigma) \text{ and } A(\alpha + \frac{\beta}{s\sigma}) = \frac{1}{|E^\sigma|} I(v^\sigma, E^\sigma) \text{ with } \sigma \in \mathbf{N}.$$

Since $|E^\sigma| = \sigma s |E^\circ|$ one gets

$$\begin{aligned} D_\beta A(\alpha) &= \lim_{\sigma\to\infty} \sigma s (A(\alpha + \frac{\beta}{\sigma s}) - A(\alpha)) = \\ &= \lim_{\sigma\to\infty} \frac{1}{|E^\circ|}(I(v^\sigma, E^\sigma) - I(u_\alpha, E^\sigma)) . \end{aligned} \tag{23}$$

We have to show that the last limit converges to $B(\alpha,\beta)$ as defined in (9). It would be favorable to know explicitly how the graphs of v^σ lie within the generalized foliation $\mathscr{M}_{\alpha,\beta}$. Since we do not know enough of the minimal solutions v^σ we replace them by a conveniently defined function w^σ with the same periodicity as v^σ. However, since w^σ needs no longer to be minimal, the price we pay is that we only obtain $D_\beta A(\alpha) \le B(\alpha,\beta)$. The reverse direction will be guaranteed by a further consideration (Lemma 5).

We define the functions w^σ interpolating between certain level functions $u_\rho = u^\sigma_\rho \in \mathscr{M}_{\alpha,\beta}$, $0 \le \rho \le \sigma$, with $u_0 = u_\alpha - 1 \le u_1 \le \dots \le u_\sigma = u_\alpha$. Let $\mathscr{P} \subseteq \mathscr{M}(\overline{a}_1, \overline{a}_2)$ be a filtration such that up to $\mathbf{Z}^{n+1}$-translations to each gap $G \in \mathscr{G}_\alpha$ there is exactly one $u_G \in \mathscr{P}$ with graph $u_G \subset G$. Recall that by definition of a foliation $\mathscr{P}$ is invariant under the $\mathbf{Z}^{n+1}$-action T. For $\rho \in \mathbf{Z}$ set

$$E^{\rho,1} \doteq \{x \in E_{\alpha,\beta} : s\rho \le x\beta \le s(\rho+1)\} = \mathrm{cl}(E^{\rho+1} \setminus E^\rho) .$$

For every $q \in \mathbf{N}$ we define level functions $u_\rho \in \mathscr{M}(\overline{a}_1) \cup \mathscr{P}$ with maximal vertical distance $\frac{1}{q}$ recursively by $u_0 \doteq u_\alpha - 1$ and

$$u_{\rho+1} \doteq \max\{u \in \mathscr{M}(\overline{a}_1) \cup \mathscr{P} : u \le u_\alpha,\ (u - u_\rho)|_{E^{\rho,1}} \le \frac{1}{q}\}$$

if $u_\rho \in \mathscr{M}(\overline{a}_1)$. In the case $u_\rho \in \mathscr{M}(\overline{a}_1, \overline{a}_2)$ we set

$$u_{\rho+1} \doteq \max\{u \in \mathscr{M}(\overline{a}_1) : u \leq u_\alpha,\ (u - u_\rho)|_{E^{\rho,1}} \leq \frac{1}{q}\} .$$

By a compactness argument in the vertical direction one shows that to every $q \in \mathbf{N}$ there is a $\sigma = \sigma(q)$ such that $u_\sigma = u_\alpha$. Moreover, $\sigma(q)$ tends to infinity if and only if q does. Finally, we construct the functions w^σ on E^σ by connecting the levels u_ρ in the direction β in an ascending order:

$$w^\sigma|_{E^{\rho,1}}(x) \doteq u_\rho(x) + \left(\frac{x\beta}{s} - \rho\right)(u_{\rho+1} - u_\rho)(x) , \quad 0 \leq \rho \leq \sigma - 1 .$$

Note that $E^\sigma = \bigcup\{E^{\rho,1} : 0 \leq \rho \leq \sigma - 1\}$.

Lemma 4 *If $\alpha \in \mathbf{Q}^n$ and $\beta = e_1 \in \mathscr{B}_\circ$ one has*

$$\lim_{\sigma \to \infty} (I(w^\sigma, E^\sigma) - I(u_\alpha, E^\sigma)) = |E^\circ| B(\alpha, \beta) .$$

Since v^σ and w^σ have the same periodicity and since v^σ is minimal, we have $I(v^\sigma, E^\sigma) \leq I(w^\sigma, E^\sigma)$. Replacing β by any standard unit vector $\pm e_i$, $1 \leq i \leq n$, we get with (23)

$$D_\beta A(\alpha) \leq B(\alpha, \beta) , \quad \alpha \in \mathbf{Q}^n ,\ \pm\beta \in \mathscr{B}_\circ .$$

The proof of the main theorem in the rational case will be completed by

Lemma 5 *If $\alpha \in \mathbf{Q}^n$, $\beta \in \mathscr{B}_\circ$, then $(D_\beta + D_{-\beta})A(\alpha) \geq B(\alpha, \beta) + B(\alpha, -\beta)$.*

Indeed, with the inequality above $D_\beta A(\alpha) = B(\alpha, \beta)$ holds for $\alpha \in \mathbf{Q}^n$, $\beta \in \mathscr{B}_\circ$. □

Proof of Lemma 4 We show that $I(w^\sigma, E^\sigma) - I(u_\alpha, E^\sigma)$ converges to $B(\alpha, \beta)$, where we interpret $B(\alpha, \beta)$ according to Corollary 3. The technique is once more to substitute

$$F(x, w^\sigma, Dw^\sigma) = F(x, u_\rho, Du_\rho) + \int_{u_\rho(x)}^{w^\sigma(x)} F_u(x, \xi, Du_\rho)\, d\xi +$$
$$+ \sum_{i=1}^{n} \frac{\partial}{\partial x^i}(w^\sigma - u_\rho) F_{p^i}(x, w^\sigma, Dw^\sigma) + O(|u_{\rho+1} - u_\rho|^2) . \qquad (24)$$

Recall that for the last term we used the estimate of Lemma 1. For each of the four summands we evaluate the limit $\sigma \to \infty$ separately.

A) We consider the contribution of the first term only of (24) to $I(w^\sigma, E^\sigma) - I(u_\alpha, E^\sigma)$. Since $I(u, E^1)$ $(= |E^1| A(\alpha))$ does not depend on the choice of the maximally periodic $u \in \mathscr{M}(\overline{a}_1)$ one gets

$$\sum_{\rho=0}^{\sigma-1}(I(u_\rho,E^{\rho,1})-I(u_\alpha,E^{\rho,1})=$$
$$=\sum_{G\in\mathcal{G}_\alpha}\ \sum_{\substack{0\le\rho\le\sigma-1\\ graph\, u_\rho\subset G}}(I(u_\rho,E^{\rho,1})-I(u_G^+,E^{\rho,1})\,.$$

Recall the notation $G=(u_G^-,u_G^+)$ where $u_G^\pm\in\mathcal{M}(\overline{a}_1)$. If for indices $\rho<\rho'$ the graphs of u_ρ and $u_{\rho'}$ are contained in the same gap G, by construction $u_\rho=u_{\rho'}=u_{G,+}$ for some $u_{G,+}\in\mathcal{P}\subseteq\mathcal{M}(\overline{a}_1,\overline{a}_2)$. Therefore, for fixed $G\in\mathcal{G}_\alpha$, the inner sum tends with $\sigma\to\infty$ to

$$\lim_{\sigma\to\infty} I(u_{G,+},E^{-\sigma}\cup E^{\sigma})-I(u_G^+,E^{-\sigma}\cup E^{\sigma})=B_G^+(E_{\alpha,\beta})\,.$$

Summing as above over all $G\in\mathcal{G}_\alpha$ one obtains

$$\begin{aligned}\lim_{\sigma\to\infty}\sum_{\rho=0}^{\sigma-1}(I(u_\rho,E^{\rho,1})-I(u_\alpha,E^{\rho,1})) &= \sum_{G\in\mathcal{G}_\alpha}B_G^+(E_{\alpha,\beta})\\ &= |E^\circ|\sum_{G\in\mathcal{G}_\alpha/\Gamma_{0,\beta}}B_G^+(E_{\alpha,\beta})\,.\end{aligned}$$

The last equality follows from the invariance of $\mathcal{M}_{\alpha,\beta}$ under $\Gamma_{0,\beta}$-translations.

B) Let the component $\alpha\beta=\frac{r}{s}$ of α be relatively prime and set $\bar{l}\doteq-(s\beta,r)$. For $0\le\rho\le\sigma$ we define the translations

$$w_\rho=w_\rho^\sigma\doteq T_{\rho\bar{l}}\,w^\sigma\quad\text{ and }\quad v_\rho\doteq T_{\rho\bar{l}}\,u_\rho\,.$$

If moreover $\triangle_\rho\doteq T_{\rho\bar{l}}\,u_{\rho+1}-T_{\rho\bar{l}}\,u_\rho$ for $x\in E^1$ one gets

$$\frac{x\beta}{s}\,\triangle_\rho(x)=w^\sigma(x)-u_\rho(x)=w_\rho(x)-v_\rho(x)\,.$$

The contribution of the middle term in (24) to $I(w^\sigma,E^\sigma)-I(u_\alpha,E^\sigma)$ takes the form

$$\sum_{\rho=0}^{\sigma-1}\int_{E^{\rho,1}}\int_{u_\rho(x)}^{w^\sigma(x)}F_u(x,\xi,Du_\rho)\,d\xi\,dx=$$
$$=\int_{E^1}\left(\sum_{\rho=0}^{\sigma-1}\int_0^{\frac{x\beta}{s}\triangle_\rho(x)}F_u(x,v_\rho+\xi,Dv_\rho)\,d\xi\right)dx\,.$$

As $\triangle_\rho(x)$ shrinks to 0 with $\sigma\to\infty$, one shows with elementary analysis that

$$\lim_{\sigma\to\infty}\sum_{\rho=0}^{\sigma-1}\int_0^{\frac{x\beta}{s}\triangle_\rho(x)}F_u(x,v_\rho+\xi,Dv_\rho)\,d\xi=\frac{x\beta}{s}\int_{\mathcal{F}_\alpha(x)}F_u(\overline{x},\Psi_\alpha)\,dx^{n+1}\,.$$

The crucial point is Moser's estimation $\|Du_{\rho+1} - Du_\rho\| \leq$ const $|u_{\rho+1} - u_\rho|$ of Lemma 1 as well as the $C^{2,\varepsilon}$-regularity of F. The convergence is even uniformly for all $x \in E^1$. Thus, in the limit $\sigma \to \infty$ the second term in of (24) contributes

$$\int_{\mathscr{F}_\alpha(E^1)} \frac{x\beta}{s} F_u(\overline{x}, \Psi_\alpha)\, d\overline{x} \, .$$

C) The contribution of the third term in (24) to $I(w^\sigma, E^\sigma) - I(u_\alpha, E^\sigma)$ is

$$\sum_{\rho=0}^{\sigma-1} \int_{E^{\rho,1}} \left(\sum_{i=1}^{n} \frac{\partial}{\partial x^i}(w^\sigma - u_\rho) F_{p^i}(x, w^\sigma, Dw^\sigma) \right) dx =$$

$$= \sum_{\rho=0}^{\sigma-1} \int_{E^1} \left(\sum_{i=1}^{n} \frac{x\beta}{s} \frac{\partial}{\partial x^i} \triangle_\rho(x) F_{p^i}(x, w_\rho, Dw_\rho) + \frac{\triangle_\rho(x)}{s} F_{p^1}(x, w_\rho, Dw_\rho) \right) dx \, .$$

Integrating the inner sum n times by part and using the periodicity on E^1, all up to one boundary evaluation cancel. Note that $w_\rho(x) = v_\rho(x) + \frac{x\beta}{s} \triangle_\rho(x)$ on E^1 and that v_ρ as well as $\triangle_\rho$ have E^1 as periodicity domain. This way, the whole expression transforms to

$$\sum_{\rho=0}^{\sigma-1} \left(\int_{E^\circ + s\beta} \triangle_\rho(x) F_{p^1}(x, v_\rho, Dv_\rho)\, dx - \int_{E^1} \frac{x\beta}{s} \triangle_\rho(x) \sum_{i=1}^{n} \frac{\partial}{\partial x^i} F_{p^i}(x, v_\rho, Dv_\rho)\, dx \right).$$

Since $F \in C^{2,\varepsilon}$ we may insert with help of Lemma 1

$$\frac{\partial}{\partial x^i} F_{p^i}(x, w_\rho, Dw_\rho)\, dx = \frac{\partial}{\partial x^i} F_{p^i}(x, v_\rho, Dv_\rho)\, dx + O(|\triangle_\rho(x)|^\varepsilon) \, .$$

Using the Euler equation $\sum \frac{\partial}{\partial x^i} F_{p^i}(x, v_\rho, Dv_\rho) = F_u(x, v_\rho, Dv_\rho)$ the second term above tends for $\sigma \to \infty$ and fixed $x \in E_1$ to

$$\lim_{\sigma \to \infty} \sum_{\rho=0}^{\sigma-1} \frac{x\beta}{s} \triangle_\rho(x)(F_u(x, v_\rho, Dv_\rho) + O(|\triangle_\rho(x)|^\varepsilon))\, dx = \int_{\mathscr{F}_\alpha(x)} \frac{x\beta}{s} F_u(\overline{x}, \Psi_\alpha) \, .$$

Because of $\sum_{\rho=0}^{\sigma-1} \triangle_\rho \leq 1$ and $\triangle_\rho \geq 0$ the error term $O(.)$ indeed contributes nothing in the limit. Therefore, the whole expression $\sum_{\rho=0}^{\sigma-1}(.)$ above tends to

$$\int_{\mathscr{F}_\alpha(E^\circ)} F_{p^1}(\overline{x}, \Psi_\alpha)\, d\overline{x} - \int_{\mathscr{F}_\alpha(E^1)} \frac{x\beta}{s} F_u(\overline{x}, \Psi_\alpha)\, d\overline{x} \, .$$

with $\sigma \to \infty$. Note that $\mathscr{F}_\alpha(E^\circ + s\beta) \equiv \mathscr{F}_\alpha(E^\circ) \bmod e_{n+1}$. Since $\mathscr{F}_\alpha(E^\circ) \bmod e_{n+1}$ actually is 1-periodic in all the standard directions $e_1, \ldots, e_n$ this expression may be rewritten by

$$|E^\circ| \int_{\mathscr{F}_\alpha(E^\circ_{0,\beta})} \beta F_p(\overline{x}, \Psi_\alpha)\, d\overline{x} - \int_{\mathscr{F}_\alpha(E^1)} \frac{x\beta}{s} F_u(\overline{x}, \Psi_\alpha)\, d\overline{x} \,.$$

Recall that $E^\circ_{0,\beta} = [0,1]^n \cap \langle\beta\rangle^\perp$ with $\beta = e_1$ and $E^\circ_{\alpha,\beta} = E^\circ$.

D) As arguing above, the estimates of $\triangle_\rho(x)$ imply that the last term in (24) contributes nothing in the limit:

$$\lim_{\sigma\to\infty} \sum_{\rho=0}^{\sigma-1} \int_{E^{\rho,1}} |(u_{\rho+1} - u_\rho)(x)|^2 dx = \lim_{\sigma\to\infty} \sum_{\rho=0}^{\sigma-1} \int_{E^1} |\triangle_\rho\,(x)|^2 dx = 0\,.$$

Adding the contributions of all four parts *A) - D)* one gets

$$\lim_{\sigma\to\infty} I(w^\sigma, E^\sigma) - I(u_\alpha, E^\sigma) =$$

$$= |E^\circ| \left(\int_{\mathscr{F}_\alpha(E^\circ_{0,\beta})} \beta F_p(\overline{x}, \Psi_\alpha)\, d\overline{x} + \sum_{G\in\mathscr{G}_\alpha/\Gamma_{0,\beta}} B^+_G(E_{\alpha,\beta}) \right) . \quad \square$$

Proof of Lemma 5 According to Corollary 3 we have to show

$$(D_\beta + D_{-\beta})A(\alpha) \geq \sum_{G\in\mathscr{G}_\alpha/\Gamma_{0,\beta}} B^+_G(E_{\alpha,\beta}) + B^-_G(E_{\alpha,\beta})\,.$$

For arbitrary three continuous functions $u_i : \mathbf{R}^n \to \mathbf{R}$ $(i = 1,2,3)$ let $\mathrm{md}(u_1, u_2, u_3) : \mathbf{R}^n \to \mathbf{R}$ be the **middle function** defined by the following requirement: If $x \in \mathbf{R}^n$ and i, j, k is a permutation of $1,2,3$ with $u_i(x) \le u_j(x) \le u_k(x)$ then $\mathrm{md}(u_1, u_2, u_3)(x) \doteq u_j(x)$.

Let us define the set

$$[u_1, u_2, u_3] \doteq \bigcup\{E^{\rho,1} : \rho \in \mathbf{Z},\, \exists x \in E^{\rho,1} \text{ with } u_1(x) \le u_2(x) \le u_3(x)\}\,.$$

Fix any enumeration of $\mathscr{G}_\alpha$. For $\sigma \in \mathbf{N}$ choose the first σ gaps in $\mathscr{G}_\alpha$. Reenumerate them by $G_i = [u_i^-, u_i^+]$, $0 \le i \le \sigma - 1$, in a way that $u_i^+ \le u_j^- \Leftrightarrow i < j$ for all $0 \le i, j \le \sigma - 1$. Without loss of generality we assume $u_0^- = u_\alpha - 1$. Set $u_\sigma^- = u_\alpha$ and choose $v^{\pm\sigma} \in \mathscr{M}^{per}_{\alpha\pm\frac{\beta}{\sigma s}}$.

We consider the variational integral of $v^{+\sigma}$ and $v^{-\sigma}$ restricted essentially to the part of $v^{\pm\sigma}$ lying within the i-th gap G_i. For each $0 \le i \le \sigma - 1$ we define

$$\Delta^\pm_\sigma(i) \doteq I(\mathrm{md}(u_i^-, v^{\pm\sigma}, u_i^+),\, [u_i^-, v^{\pm\sigma}, u_i^+]) - \mathrm{vol}_n[u_i^-, v^{\pm\sigma}, u_i^+] \cdot A(\alpha)\,.$$

Note that $\mathrm{vol}_n[u_i^-, v^{\pm\sigma}, u_i^+]$ is an integer multiple of $\mathrm{vol}_n(E^1)$. Loosely speaking, $\Delta^+_\sigma(i)$ is the additional amount of the variational integral needed to join the bottom u_i^- of the gap G_i with its top u_i^+ along $v^{+\sigma}$. This amount is bounded from below by $B^+_{G_i}(E_{\alpha,\beta})$. The sum may be estimated by

$$\Delta^+_\sigma(i) + \Delta^-_\sigma(i) \geq B^+_{G_i}(E_{\alpha,\beta}) + B^-_{G_i}(E_{\alpha,\beta})\,. \tag{25}$$

To check this inequality more detailed, one has to apply the standard reasoning as it is described in the proof of Theorem 2, part one.

For the part of $v^{+\sigma}$ and $v^{-\sigma}$, respectively, lying between two successive gaps G_i and G_{i+1} we define similarly

$$\Delta_\sigma^\pm(i,i+1) \doteq I(\mathrm{md}(u_i^+, v^{\pm\sigma}, u_{i+1}^-),\ [u_i^+, v^{\pm\sigma}, u_{i+1}^-]) - \mathrm{vol}_n[u_i^+, v^{\pm\sigma}, u_{i+1}^-]\cdot A(\alpha)\,.$$

Using the periodicity and minimality of u_i^- and u_i^+, one shows with analogous standard arguments as above

$$\Delta_\sigma^+(i,i+1) + \Delta_\sigma^-(i,i+1) \geq 0\,. \tag{26}$$

The next equality is just a reordering of the various summands:

$$\begin{aligned}\sum_{i=0}^{\sigma-1}&(\Delta_\sigma^\pm(i) + \Delta_\sigma^\pm(i,i+1)) = \\ &= I(v^{\pm\sigma},\ [u_\alpha - 1, v^{\pm\sigma}, u_\alpha]) - \mathrm{vol}_n[u_\alpha - 1, v^{\pm\sigma}, u_\alpha]\cdot A(\alpha) = \\ &= I(v^{\pm\sigma}, E^{\pm\sigma}) - I(u_\alpha, E^{\pm\sigma})\,.\end{aligned}$$

Note that those parts of the variational integral which were counting twice, one at times cancel with a term $\mathrm{vol}_n(E^{\rho,1})\cdot A(\alpha)$.

Finally, we add the two versions $\pm$ of the last equality. Inserting (25) and (26) we get

$$\begin{aligned}I(v^{+\sigma}, E^\sigma) - I(u_\alpha, E^\sigma) + I(v^{-\sigma}, E^{-\sigma}) - I(u_\alpha, E^{-\sigma}) &\geq \\ \geq \sum_{i=0}^{\sigma-1} B_{G_i}^+(E_{\alpha,\beta}) + B_{G_i}^-(E_{\alpha,\beta})\,.&\end{aligned}$$

In the limit $\sigma \to \infty$ we obtain with (23) the desired inequality

$$\begin{aligned}|E^\circ|(D_\beta A(\alpha) + D_{-\beta}A(\alpha)) &\geq \sum_{G\in\mathcal{G}_\alpha} B_G^+(E_{\alpha,\beta}) + B_G^-(E_{\alpha,\beta}) = \\ &= |E^\circ| \sum_{G\in\mathcal{G}_\alpha/\Gamma_{0,\beta}} B_G^+(E_{\alpha,\beta}) + B_G^-(E_{\alpha,\beta})\,. \quad\square\end{aligned}$$

5.2 Limits of generalized foliations

In order to establish continuity properties of $B(\alpha,\beta)$ in α, one first has to prove this properties for the corresponding sets $\mathcal{M}_{\alpha,\beta}$. We show that for any $\alpha\in\mathbf{R}^n$ and $\beta\in(V_\alpha\cup V_\alpha^\perp)\cap S_Q^{n-1}$ there is an approximating sequence $\alpha'\to\alpha$ such that $\lim_{\alpha'\to\alpha}\mathcal{M}_{\alpha',\beta} = \mathcal{M}_{\alpha,\beta}$ and such that moreover $\lim_{\alpha'\to\alpha}\frac{\alpha'-\alpha}{\|\alpha'-\alpha\|} = \beta$ is satisfied. Recall that the latter condition will be used to ensure the convergence of the corresponding derivatives $D_\beta A(\alpha')$ to $D_\beta A(\alpha)$.

We expand the definition of $\mathcal{M}_{\alpha,\beta}$ in Sect. 3.2 by setting $\mathcal{M}_{\alpha,\beta}\doteq\mathcal{M}(\overline{a}_1)$ in the case $\beta=0$ as well as in the case $\beta\in S_Q^{n-1}$ with $\beta\notin V_\alpha$. By $\lim_{\alpha'\to\alpha}\mathcal{M}_{\alpha',\beta}$ we denote the set

$$\lim_{\alpha'\to\alpha} \mathscr{M}_{\alpha',\beta} \doteq \{u \in W^{1,2}_{loc}(\mathbf{R}^n) : \forall \alpha' \, \exists u_{\alpha'} \in \mathscr{M}_{\alpha',\beta} \text{ such that } \lim_{\alpha'\to\alpha} u_{\alpha'} = u\} ,$$

where the limit $\lim u_{\alpha'}$ is understood with respect to the C^1-topology on compact sets.

Lemma 6 *Let $\alpha \in \mathbf{R}^n$ be arbitrary and $\beta \in (V_\alpha \cup V_\alpha^\perp) \cap S_Q^{n-1}$ or $\beta = 0$. If $\alpha' \to \alpha$ with $\alpha' \in \mathbf{Q}^n$ is any sequence satisfying $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$ for all α' then*

$$\lim_{\alpha'\to\alpha} \mathscr{M}_{\alpha',\beta} = \mathscr{M}_{\alpha,\beta}$$

Proof First we show $\lim_{\alpha'\to\alpha} \mathscr{M}_{\alpha',\beta} \subseteq \mathscr{M}_{\alpha,\beta}$. Assume that $\beta \in V_\alpha$. Since $V_{\alpha'} = \mathbf{R}^n$ for $\alpha' \in \mathbf{Q}^n$, the set $\mathscr{M}_{\alpha',\beta}$ is well defined. Recall that $\overline{\Gamma}_{\alpha,\beta} = \{\overline{k} = (k, k^{n+1}) \in \overline{\Gamma}_\alpha : k\beta = 0\}$. From $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$ one concludes $\overline{\Gamma}_{\alpha,\beta} \subseteq \overline{\Gamma}_{\alpha',\beta}$ and since moreover $\overline{\Gamma}_{\alpha,\beta} = \overline{\Gamma}_\alpha \cap \overline{\Gamma}_{\alpha',\beta}$ it holds $\overline{\Gamma}_\alpha \setminus \overline{\Gamma}_{\alpha,\beta} \subseteq \overline{\Gamma}_{\alpha'} \setminus \overline{\Gamma}_{\alpha',\beta}$. If $\beta \in V_\alpha^\perp$ or $\beta = 0$, by definition $\overline{\Gamma}_{\alpha,\beta} = \overline{\Gamma}_\alpha$ and both inclusions again are true. Therefore, in any case $u \in \mathscr{M}_{\alpha',\beta}$ satisfies

$$T_{\overline{k}} u = u \ \forall \overline{k} \in \overline{\Gamma}_{\alpha,\beta} \quad \text{and} \quad T_{\overline{k}} u \geq u \ \forall \overline{k} \in \overline{\Gamma}_\alpha \setminus \overline{\Gamma}_{\alpha,\beta} \text{ with } \overline{k}\overline{a}_2 \geq 0 . \tag{27}$$

Recall the definition of $\overline{a}_2$ in sect. 2.2.

Moreover, every $u \in \lim_{\alpha'\to\alpha} \mathscr{M}_{\alpha',\beta}$ again satisfies this property. Since by definitions $\mathscr{M}_{\alpha,\beta}$ consists of all $u \in \mathscr{M}_\alpha$ with property (27), one concludes that $\lim_{\alpha'\to\alpha} \mathscr{M}_{\alpha',\beta} \subseteq \mathscr{M}_{\alpha,\beta}$.

To establish the converse inclusion, we have to approximate a given $v \in \mathscr{M}_{\alpha,\beta}$ by solutions $u_{\alpha'} \in \mathscr{M}_{\alpha',\beta}$. Assume $\alpha' \neq \alpha$. Due to the intermediate value theorem and the $\mathbf{Z}^{n+1}$-periodicity of $\mathscr{M}_{\alpha',\beta}$, there exist $x_{\alpha'} \in [0,1]^n$ and $u_{\alpha'} \in \mathscr{M}_{\alpha',\beta}$ with $u_{\alpha'}(x_{\alpha'}) = v(x_{\alpha'})$. By Moser's compactness theorem [15, Cor. 3.3], there is subsequence $\alpha' \to \alpha$ such that $\lim_{\alpha'\to\alpha} u_{\alpha'}$ exists and corresponds to some $u \in \mathscr{M}_\alpha$. The first part of the proof states that even $u \in \mathscr{M}_{\alpha,\beta}$. Moreover, u and v coincide at every accumulation point of $x_{\alpha'} \in [0,1]^n$. Since the set $\mathscr{M}_{\alpha,\beta}$ is totally ordered [5, (6.22)], we conclude $u = v$. The C^1-convergence is due to the regularity properties stated in Lemma 1. □

We want to weaken the condition $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$ in Lemma 6. If $\beta \in V_\alpha^\perp \cap S_Q^{n-1}$ one finds a sequence $\alpha' \to \alpha$ satisfying (21) as well as $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$. The second condition is equivalent to $\overline{a}_1' \doteq \frac{(-\alpha',1)}{\|(-\alpha',1)\|} \in \overline{V}_\alpha^\perp$, where $\overline{V}_\alpha^\perp$ is the orthogonal complement of $\overline{V}_\alpha = \operatorname{span}_{\mathbf{R}} \overline{\Gamma}_\alpha$ within $\mathbf{R}^{n+1}$. Note that by assumption on β we have $\overline{a}_2 \in \overline{V}_\alpha^\perp$ too. Property (21) states that the sequence $\overline{a}_1' \to \overline{a}_1$ has to satisfy $\lim \frac{\overline{a}_1' - \overline{a}_1}{\|\overline{a}_1' - \overline{a}_1\|} = \overline{a}_2$. If now $\beta \in V_\alpha \cap S_Q^{n-1}$, i.e. $\overline{a}_2 \in \overline{V}_\alpha \cap S_Q^n$, a sequence $\overline{a}_1' \to \overline{a}_1 \in \overline{V}_\alpha^\perp$ with (21) and $\overline{a}_1' \in \overline{V}_\alpha^\perp$ (and thus $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$) may no longer be found. For this case, Lemma 6 is modified in the following way:

Lemma 7 *Suppose $\alpha \in \mathbf{R}^n$ and $\beta \in V_\alpha \cap S_Q^{n-1}$. Let $\alpha' \to \alpha$, $\alpha' \in \mathbf{Q}^n$, satisfy*

$$\overline{\Gamma}_{\alpha,\beta} \subseteq \overline{\Gamma}_{\alpha'} \quad \textit{and} \quad \overline{a}_1' \cdot \overline{a}_2 > 0 . \tag{28}$$

Then $\lim_{\alpha'\to\alpha} \mathscr{M}_{\alpha',\beta} = \mathscr{M}_{\alpha,\beta}$.

Remark If $\beta = e_i$ the condition $\overline{a}'_1 \cdot \overline{a}_2 > 0$ is equivalent to $(\alpha')^i > \alpha^i$. □

According to condition (28), the vector $\overline{a}'_1$ may be varied within the orthogonal complement of $\overline{\Gamma}_{\alpha,\beta}$ and such that it lies within the half space $\{\overline{x} \in \mathbf{R}^{n+1} : \overline{x}\overline{a}_2 > 0\}$. Note that $\overline{a}_2$ is orthogonal to $\overline{\Gamma}_{\alpha,\beta}$ too. Thus, if $\beta \in V_\alpha \cap S_Q^{n-1}$ one finds a sequence $\alpha' \to \alpha$ with property (21) as well as (28). The lemma states $\lim_{\alpha' \to \alpha} \mathscr{M}_{\alpha',\beta} = \mathscr{M}_{\alpha,\beta}$ for this sequence again.

To prove the lemma we rewrite the condition (28) in the somewhat complicated manner

$$\overline{\Gamma}_{\alpha,\beta} \subseteq \overline{\Gamma}_{\alpha'} \quad \text{and} \quad \overline{k}\overline{a}'_1 > 0 \text{ if } \overline{k} \in \overline{\Gamma}_\alpha \setminus \overline{\Gamma}_{\alpha,\beta} \text{ with } \overline{k}\overline{a}_2 > 0 .$$

One concludes that for α' having this property any $u \in \mathscr{M}_{\alpha',\beta}$ has to satisfy the key-property (27) of the preceding proof. The proof is therefore valid for Lemma 7 as well.

5.3 Continuity properties of $D_\beta A(\,.\,)$

We establish the main theorem for the remaining case $\alpha \in \mathbf{R}^n \setminus \mathbf{Q}^n$. As a general fact for convex functions we need

Lemma 8 *Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex, $\alpha \in \mathbf{R}^n$ and $\beta \in S_Q^{n-1}$. For any sequence $\alpha' \to \alpha$ with $\alpha' \neq \alpha$ and $\lim_{\alpha' \to \alpha} \frac{\alpha' - \alpha}{\|\alpha' - \alpha\|} = \beta$ one has*

$$\lim_{\alpha' \to \alpha} D_\beta f(\alpha') = D_\beta f(\alpha) .$$

Apart from the convexity, the proof of Lemma 8 uses the local Lipschitz-continuity of f [19, thm 10.4] and the upper semi-continuity of $D_\beta f(\,.\,)$, see [19, cor 24.5.1 and thm 24.6].

We show in a indirect way that $B(\,.\,,\beta)$ is continuous with respect to a particular sequence $\alpha' \to \alpha$ as well.

Lemma 9 *Let $\mathscr{B}_\circ$ be α-admissible, $\alpha \in \mathbf{R}^n$ and $\beta \in \mathscr{B}_\circ$. Let the approximating sequence $\alpha' \to \alpha$, $\alpha' \in \mathbf{Q}^n$, satisfy $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$ in the case $\beta \in \mathscr{B}_\circ \cap V_\alpha^\perp$ respectively* (28) *in the case $\beta \in \mathscr{B}_\circ \cap V_\alpha$. Then one has*

$$\lim_{\alpha' \to \alpha} B(\alpha', \beta) \leq B(\alpha, \beta) .$$

The proof is postponed to the end of the section.

Now we use that one already knows $D_\beta A(\alpha') = B(\alpha', \beta)$ for $\alpha' \in \mathbf{Q}^n$ and $\beta \in \mathscr{B}_\circ$. Let $\alpha' \to \alpha$ be a sequence with $\alpha' \in \mathbf{Q}^n$ and $\lim_{\alpha' \to \alpha} \frac{\alpha' - \alpha}{\|\alpha' - \alpha\|} = \beta$. Suppose it moreover satisfies $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$ or (28), respectively. The existence of such a sequence is discussed in the preceding section. Combining the last two lemmas we get

$$D_\beta A(\alpha) \leq B(\alpha, \beta) \quad \text{if } \mathscr{B}_\circ \subseteq V_\alpha \cup V_\alpha^\perp ,\ \beta \in \mathscr{B}_\circ . \tag{29}$$

Equality will be established by

Lemma 10 *Let $\mathscr{B}_\circ$ be α-admissible, $\alpha \in \mathbf{R}^n$ and $\beta \in \mathscr{B}_\circ \cap V_\alpha$. If $\alpha' \to \alpha$, $\alpha' \in \mathbf{R}^n$, is an approximating sequence with $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$ then*

$$\lim_{\alpha' \to \alpha} B(\alpha', \beta) + B(\alpha' - \beta) \geq B(\alpha, \beta) + B(\alpha, -\beta) .$$

(If instead of $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$ we only assume (28), the statement of the lemma is no longer true in general.)

Notice that for $\beta \in \mathscr{B}_\circ \cap V_\alpha^\perp$, the inequality of Lemma 10 trivially holds for any sequence $\alpha' \to \alpha$ with $\alpha \in \mathbf{Q}^n$: by (10) one has $B(\alpha, \beta) + B(\alpha, -\beta) = 0$ while by convexity $B(\alpha', \beta) + B(\alpha', -\beta) = (D_\beta + D_{-\beta})A(\alpha') \geq 0$.

Before proving the lemmas we deduce the main theorem and some further result. We again use that for $\alpha' \in \mathbf{Q}^n$ we already showed $B(\alpha', \pm\beta) = D_{\pm\beta}A(\alpha')$ with $\alpha' \in \mathbf{Q}^n$. From Lemma 10 and the notice above it follows that

$$(D_\beta + D_{-\beta})A(\alpha) \geq B(\alpha, \beta) + B(\alpha, -\beta) ,$$

with any $\beta \in \mathscr{B}_\circ$. Here we made use of the upper semi-continuity of $D_\beta A(\,.\,)$ In view of (29), this proves the main theorem:

$$\text{If } \alpha \in \mathbf{R}^n \text{ with } \mathscr{B}_\circ \subseteq V_\alpha \cup V_\alpha^\perp \text{ then } D_\beta A(\alpha) = B(\alpha, \beta) \quad \forall\, \beta \in \mathscr{B}_\circ . \quad \square \tag{30}$$

Finally, we summarize the limit behavior of $B(\,.\,, \beta)$ and formulate it as continuity property of $D_\beta A(\,.\,)$. Actually, we have established the limit behavior of $B(\,.\,, \beta)$ for a different class of sequences $\alpha' \to \alpha$ than occuring in Lemma 8. This circumstance may be converted into a stronger continuity property for $D_\beta A(\,.\,)$.

Theorem 4 *Let $\alpha \in \mathbf{R}^n$ and $\alpha' \to \alpha$, $\alpha' \in \mathbf{R}^n$, be an approximating sequence with $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$ for all α'. Then for all $\beta \in V_\alpha \cup V_\alpha^\perp$*

$$\lim_{\alpha' \to \alpha} D_\beta A(\alpha') = D_\beta A(\alpha) . \tag{31}$$

In the case $\beta \in V_\alpha^\perp$, (31) is true for any sequence $\alpha' \to \alpha$, $\alpha' \in \mathbf{R}^n$.

The theorem generalizes the fact that $A|_D$ is continuously differentiable at a rationally independent point $\alpha \in \mathbf{R}^n$, D denoting the set of points where A is differentiable.

Proof Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex and $(D_\beta + D_{-\beta})f(\alpha) = 0$. From the lower semi-continuity of $D_\beta f(.)$ one concludes $\lim_{\alpha' \to \alpha} D_\beta f(\alpha') = D_\beta f(\alpha)$ for any sequence $\alpha' \to \alpha$, $\alpha' \in \mathbf{R}^n$. The second part of the theorem therefore follows from Theorem 1 and 2. The first part is proven with the same idea:

We may assume that $\mathscr{B}_\circ$ is α-admissible. Let $\alpha' \to \alpha$ satisfy $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma}_{\alpha'}$. Using (30), Lemma 10 states

$$\lim_{\alpha' \to \alpha} (D_\beta + D_{-\beta})A(\alpha') \geq B(\alpha, \beta) + B(\alpha, -\beta) .$$

But the upper semi-continuity and again (30) yield

$$\lim_{\alpha' \to \alpha} D_{\pm\beta} A(\alpha') \leq D_{\pm\beta} A(\alpha) = B(\alpha, \pm\beta) .$$

Combining this two inequalities, the proof will be completed. □

Proof of Lemma 9 We use the same technique as in the proof of Lemma 3 and 4. The idea is to replace $B(\alpha', \beta)$ by a quantity $B^{\varepsilon}(\alpha', \beta)$ for which we know the limit behavior with respect to $\alpha' \to \alpha$. We first show, roughly speaking, that $B^{\varepsilon}(\alpha', \beta)$ is greater than $B(\alpha', \beta)$ and second, that $B^{\varepsilon}(\alpha', \beta)$ converges to $B(\alpha, \beta)$ as $\alpha' \to \alpha$. This proves the lemma. Some preliminary definitions are needed.

Suppose, exactly the first r components of α are rational, thus E_α having the form (7). If $\beta \in \mathscr{B}_\circ \cap V_\alpha$, say $\beta = e_r$, then

$$E^+_{\alpha,\beta} = \text{span}_{[0,1]}\{s^1 e_1, \ldots, s^{r-1} e_{r-1}\} \times \text{span}_{\mathbf{R}^+}\{e_r\} \times \text{span}_{\mathbf{R}}\{e_{r+1}, \ldots, e_n\} .$$

If $\beta \in \mathscr{B}_\circ \cap V_\alpha^\perp$, say $\beta = e_{r+1}$, $E^+_{\alpha,\beta}$ has the same form with r replaced by $r+1$. We restrict ourselves to the case $\beta = e_r \in \mathscr{B}_\circ \cap V_\alpha$, the other case being similar. For $\tau \in \mathbf{N}$ we define the subset

$$Q_\tau \doteq \text{span}_{[0,1]}\{s^1 e_1, \ldots, s^{r-1} e_{r-1}\} \times \text{span}_{[0,\tau]}\{e_r\} \times \text{span}_{[-\tau,\tau]}\{e_{r+1}, \ldots, e_n\} .$$

of $E^+_{\alpha,\beta}$. Set $Q^\circ_\tau \doteq Q_\tau \cap \langle\beta\rangle^\perp$. Fix an arbitrary $\varepsilon > 0$ and choose $\tau = \tau(\varepsilon) \in \mathbf{N}$ such that

$$\sum_{G \in \mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} \text{vol}_n\{\overline{x} \in G : x \in E^\circ_{\alpha,\beta} \setminus Q^\circ_\tau \text{ or } x \in \text{bd}(Q_\tau) \setminus Q^\circ_\tau\} \leq \varepsilon . \tag{32}$$

This is possible since $\sum \text{vol}_n\{\overline{x} \in G : x \in E^\circ_{\alpha,\beta}\}$ and $\sum \text{vol}_{n+1}\{\overline{x} \in G : x \in E^+_{\alpha,\beta}\}$ is less than 1 when summing up over all $G \in \mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}$.

Next, we approximate the gaps in a fixed system of representatives $\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}$ by those of $\mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}$. Note that, according to the preceding section $\lim_{\alpha' \to \alpha} \mathscr{M}_{\alpha',\beta} = \mathscr{M}_{\alpha,\beta}$. To every α' of the sequence and every $G \in \mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}$ choose $G'_G \in \mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}$ such that for an appropriate $\overline{k}'_G \in \Gamma_{0,\beta} \times \mathbf{Z}$ the translate $G'_G - \overline{k}'_G$ best approximates the gap G (with respect to the Hausdorff metric) when restricted to the compact set $G \cap (Q_\tau \times \mathbf{R}) \subset \mathbf{R}^{n+1}$. In particular, one has $\lim_{\alpha' \to \alpha}(G'_G - \overline{k}'_G) = G$. It may be that the same G' is needed to approximate different $G_1 \neq G_2$ in $\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}$. For a given $G' \in \mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}$ let $\mathscr{G}_{G'}$ be the set of all $G \in \mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}$ with $G' = G'_G$. Define

$$V_{G'} \doteq \bigcup_{G \in \mathscr{G}_{G'}} k'_G + Q_\tau \subseteq E^+_{\alpha,\beta} ,$$

where $\overline{k}'_G = (k'_G, (k'_G)^{n+1})$ is as above. If $\mathscr{G}_{G'} = \emptyset$ put $V_{G'} = \emptyset$. Put $V^\circ_{G'} \doteq V_{G'} \cap \langle\beta\rangle^\perp$. Since the gaps in $\mathscr{G}_{G'}$ are pairwise not equivalent with respect to $\overset{\Gamma_{0,\beta}}{\sim}$, the union defining $V_{G'}$ is disjoint for α' sufficiently close to α and fixed τ.

Now, we are ready to define the quantity $B^{\varepsilon}(\alpha',\beta)$ approximating $B(\alpha,\beta)$. For any $G' \in \mathscr{G}_{\alpha',\beta}$ we define the function $v_{G'}$ on $E^{+}_{\alpha',\beta}$ by

$$v_{G'}(x) \doteq \begin{cases} u^{-}_{G'}(x) + x\beta(u^{+}_{G'} - u^{-}_{G'})(x) & \text{if } x\beta \leq 1\ , \\ u^{+}_{G'}(x) & \text{if } x\beta > 1\ . \end{cases}$$

By assumption on α' we have $E^{+}_{\alpha',\beta} \subseteq E^{+}_{\alpha,\beta}$. Recall $G' = (u^{-}_{G'}, u^{+}_{G'})$. For $\Omega \subseteq \mathbf{R}^n$ we abbreviate

$$B^{1}_{G'}(\Omega) \doteq \int_{\Omega} (F(x, v_{G'}, Dv_{G'}) - F(x, u^{+}_{G'}, Du^{+}_{G'}))\, dx\ .$$

The expression $B^{\varepsilon}(\alpha',\beta)$ is defined by

$$B^{\varepsilon}(\alpha',\beta) \doteq \int_{\mathscr{F}_{\alpha',\beta}(E^{\circ}_{0,\beta})} \beta F_p(\overline{x}, \Psi_{\alpha',\beta})\, d\overline{x} \;+ \sum_{G' \in \mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}} B^{1}_{G'}(E^{+}_{\alpha',\beta} \setminus V_{G'}) + \\ + \sum_{G' \in \mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}} B_{G'}(V_{G'})\ .$$

A) First we estimate $B^{\varepsilon}(\alpha',\beta)$ from below with help of $B(\alpha',\beta)$. For two sets G_1, $G_2 \in \mathbf{R}^n$ we denote the symmetric difference by $G_1 \triangle G_2$. Since the convergence $\lim_{\alpha' \to \alpha} \mathscr{M}_{\alpha',\beta} = \mathscr{M}_{\alpha,\beta}$ is C^1 one has with arbitrary fixed τ

$$\sum_{G \in \mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} \mathrm{vol}_n \Big(\{\overline{x} \in G : x \in \mathrm{bd}(Q_\tau)\} \triangle \\ \triangle \{\overline{x} \in G'_G - \overline{k}'_G : x \in \mathrm{bd}(Q_\tau)\} \Big) \leq \varepsilon$$

for α' close enough to α. From (32) we in particular conclude

$$\sum_{G' \in \mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}} \mathrm{vol}_n \{\overline{x} \in G' : x \in \mathrm{bd}(V_{G'}) \setminus V^{\circ}_{G'}\} \leq 2\varepsilon\ . \tag{33}$$

Using the minimality of $u^{-}_{G'}$ one by the standard reasoning shows $B^{1}_{G'}(E^{+}_{\alpha',\beta}) \geq B_{G'}(E^{+}_{\alpha',\beta})$. (Compare also the explanation to formula (35) below.) Taking into account the estimate (33), one gets with some little modification of the arguments

$$\sum_{G' \in \mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}} B^{1}_{G'}(E^{+}_{\alpha',\beta} \setminus V_{G'}) \geq \sum_{G' \in \mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}} B_{G'}(E^{+}_{\alpha',\beta} \setminus V_{G'}) - 2\varepsilon \cdot \mathrm{const}\ .$$

The constant only depends on F and some compact set containing α in its interior. This shows that for α' close to α

$$B^{\varepsilon}(\alpha',\beta) \geq B(\alpha',\beta) - 2\varepsilon \cdot \mathrm{const}\ .$$

B) Second, we establish the convergence of $B^{\varepsilon}(\alpha',\beta)$ to $B(\alpha,\beta)$ with respect to the limit $\alpha' \to \alpha$. More precisely, we show

$$|\lim_{\alpha'\to\alpha} B^\varepsilon(\alpha',\beta) - B(\alpha,\beta)| \le 5\,\varepsilon\cdot\text{const}\,,$$

where the constant depends again only on F and some compact set containing α. Since $\varepsilon > 0$ is arbitrary, this together with part A) proves the lemma.

1. We claim that the last term of $B^\varepsilon(\alpha',\beta)$ may be estimated by

$$\left|\sum_{G'\in\mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}} B_{G'}(V_{G'}) - \sum_{G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} B_G(E^+_{\alpha,\beta})\right| \le 2\,\varepsilon\cdot\text{const}\,, \tag{34}$$

if α' is sufficiently close to α. By definition we have

$$\sum_{G'\in\mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}} B_{G'}(V_{G'}) = \sum_{G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} B_{G'_G-\overline{k}'_G}(Q_\tau)\,.$$

Since $\mathscr{F}_{\alpha,\beta}(Q_\tau)$ is compact and the convergence $\mathscr{M}_{\alpha',\beta}\to\mathscr{M}_{\alpha,\beta}$ is C^1 (sect. 5.2), the right hand side converges to $\sum_{G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} B_G(Q_\tau)$. Therefore, estimate (34) follows from

$$\left|\sum_{G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} B_G(Q_\tau) - \sum_{G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} B_G(E^+_{\alpha,\beta})\right| \le \varepsilon\cdot\text{const}\,. \tag{35}$$

The proof of (35) uses the standard reasoning (see sect. 4.1) which will be briefly outlined again. Let $B_R \subseteq \mathbf{R}^n$ be the ball of radius $R > 0$ and center at the origin. On $(E^+_{\alpha,\beta}\setminus Q_\tau)\cap B_R$ we construct with help of u^+_G compact variations of u^-_G and vice versa. Since both, u^-_G and u^+_G are minimal, we get from (32)

$$\sum_{G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}} |B_G(E^+_{\alpha,\beta}\setminus Q_\tau)| \le \varepsilon\cdot\text{const}$$

by taking the limit $R\to\infty$. The arguments rely on the periodicity of u^-_G and u^+_G in the directions $e_1,\dots,e_{r-1}$ as well as their asymptoticity in the remaining directions $e_r,\dots,e_n$. In order to apply the minimality, we use the fact that $\sum \text{vol}_n\{\overline{x}\in G : x\in E^+_{\alpha,\beta}\cap\text{bd}(B_R)\}$, when summing up over all $G\in\mathscr{G}_{\alpha,\beta}/\Gamma_{0,\beta}$, tends to 0 for a sequence $R\to\infty$. (Compare the proof of Theorem 2, part one.) Formula (35) is just the complementary formulation of the estimate above.

2. Considering the definition of $B^\varepsilon(\alpha',\beta)$ and $B(\alpha,\beta)$ it remains to show that

$$\left|\lim_{\alpha'\to\alpha}\left(\int_{\mathscr{F}_{\alpha',\beta}(E^\circ_{0,\beta})}\beta F_p(\overline{x},\Psi_{\alpha',\beta})\,d\overline{x} + \sum_{G'\in\mathscr{G}_{\alpha',\beta}/\Gamma_{0,\beta}} B^1_{G'}(E^+_{\alpha',\beta}\setminus V_{G'})\right) - \right.$$
$$\left. - \int_{\mathscr{F}_{\alpha,\beta}(E^\circ_{0,\beta})}\beta F_p(\overline{x},\Psi_{\alpha,\beta})\,d\overline{x}\right| \le 3\,\varepsilon\cdot\text{const}\,. \tag{36}$$

As in the proof of formula (19), we integrate $B^1_{G'}(E^+_{\alpha',\beta} \setminus V_{G'})$ n-times by part after substituting (18). Ignoring the gap $V_{G'}$ in the domain of integration we get

$$B^1_{G'}(E^+_{\alpha',\beta}) = \int_{E^\circ_{\alpha',\beta}} (u^+_{G'} - u^-_{G'})\beta F_p(x, u^+_{G'}, p_{G'})\, dx \ ,$$

where $p_{G'}(x) \in \mathbf{R}^n$ are middle values satisfying $\|(p_{G'} - Du^+_{G'})(x)\| \leq \text{const} \cdot (u^+_{G'} - u^-_{G'})(x)$. Hereby, all up to the remaining term cancel due to the periodicity on $E^\circ_{\alpha',\beta}$ of the functions involved. Taking into account the gap $V_{G'}$, we have to correct this equality to

$$\left| B^1_{G'}(E^+_{\alpha',\beta} \setminus V_{G'}) - \int_{E^\circ_{\alpha',\beta} \setminus V^\circ_{G'}} (u^+_{G'} - u^-_{G'})\beta F_p(x, u^+_{G'}, p_{G'})\, dx \right| \leq$$
$$\leq \text{const} \cdot \text{vol}_n \{\overline{x} \in G' : x \in \text{bd}(V_{G'}) \setminus V^\circ_{G'}\} \ .$$

The term $\sum B^1_{G'}(E^+_{\alpha',\beta} \setminus V_{G'})$ in (36) may therefore be replaced by

$$\sum_{G' \in \mathcal{G}_{\alpha',\beta}/\Gamma_{0,\beta}} \int_{E^\circ_{\alpha',\beta} \setminus V^\circ_{G'}} (u^+_{G'} - u^-_{G'})\beta F_p(x, u^+_{G'}, p_{G'})\, dx \tag{37}$$

with an error term bounded by $2\varepsilon \cdot \text{const}$ according to (33). We decompose this sum in a sum of integrals each of it with integration domain restricted to $E^\circ_{0,\beta}$.

Consider the set

$$\mathcal{G}^{\circ,\varepsilon}_{\alpha',\beta} \doteq \{G' \in \mathcal{G}_{\alpha',\beta} : G' = G'_G - \overline{k} \text{ for some } G \in \mathcal{G}_{\alpha,\beta}/\Gamma_{0,\beta} \text{ and } \overline{k} \in \mathbf{Z}^{n+1}\}$$

and put $\mathcal{G}^{1,\varepsilon}_{\alpha',\beta} \doteq \mathcal{G}_{\alpha',\beta} \setminus \mathcal{G}^{\circ,\varepsilon}_{\alpha',\beta}$. Although for $\alpha' \to \alpha$ each gap in $\mathcal{G}^1_{\alpha',\beta}$ will collaps, their union may furnish in the limit a subset of $\mathcal{F}_{\alpha,\beta}(E_{0,\beta}) \subseteq \mathbf{R}^{n+1}$ of positive measure. We have to take into account its influence to the limit $\lim_{\alpha' \to \alpha} B^\varepsilon(\alpha', \beta)$.

For α' close to α we have

$$\bigcup_{G' \in \mathcal{G}_{\alpha',\beta}/\Gamma_{0,\beta}} \{\overline{x} \in G' : x \in V^\circ_{G'}\} \equiv \bigcup_{G' \in \mathcal{G}^{\circ,\varepsilon}_{\alpha',\beta}} \{\overline{x} \in G' : x \in E^\circ_{0,\beta}\} \ \text{mod}\mathbf{Z}^{n+1}$$

by definition of $V_{G'}$. Taking on both sides the complement within the set $\bigcup_{G' \in \mathcal{G}_{\alpha',\beta}} \{\overline{x} \in G' : x \in E^\circ_{0,\beta}\} \ \text{mod}\mathbf{Z}^{n+1}$ one obtains

$$\bigcup_{G' \in \mathcal{G}_{\alpha',\beta}/\Gamma_{0,\beta}} \{\overline{x} \in G' : x \in E^\circ_{\alpha',\beta} \setminus V^\circ_{G'}\} \equiv$$
$$\equiv \bigcup_{G' \in \mathcal{G}^{1,\varepsilon}_{\alpha',\beta}} \{\overline{x} \in G' : x \in E^\circ_{0,\beta}\} \ \text{mod}\mathbf{Z}^{n+1} \ .$$

By $A \triangle B$ we denote the symmetric difference of the sets A and B. If $G \in \mathcal{G}_{\alpha,\beta}$ choose $G'(G) \in \mathcal{G}_{\alpha',\beta}$ such that the n-dimensional volume of $\{\overline{x} \in G'(G) : x \in E^\circ_{0,\beta}\} \triangle \{\overline{x} \in G : x \in E^\circ_{0,\beta}\}$ is minimal. Defining

$$\mathcal{G}^1_{\alpha',\beta} \doteq \{G' \in \mathcal{G}_{\alpha',\beta} : G' \neq G'(G) \text{ for all } G \in \mathcal{G}_{\alpha,\beta}\}$$

one has according to (32) for α' close to α

$$\mathrm{vol}_n\left(\bigcup_{G' \in \mathcal{G}^{1;\varepsilon}_{\alpha',\beta}} \{\overline{x} \in G' : x \in E^\circ_{0,\beta}\} \triangle \bigcup_{G' \in \mathcal{G}^1_{\alpha',\beta}} \{\overline{x} \in G' : x \in E^\circ_{0,\beta}\}\right) \leq \varepsilon .$$

From the last identity $\mathrm{mod}\mathbf{Z}^{n+1}$ above it follows

$$\mathrm{vol}_n\left(\bigcup_{G' \in \mathcal{G}_{\alpha',\beta}/\Gamma_{0,\beta}} \{\overline{x} \in G' : x \in E^\circ_{\alpha',\beta} \setminus V^\circ_{G'}\}\mathrm{mod}\mathbf{Z}^{n+1} \triangle \right.$$

$$\left. \triangle \bigcup_{G' \in \mathcal{G}^1_{\alpha',\beta}} \{\overline{x} \in G' : x \in E^\circ_{0,\beta}\}\mathrm{mod}\mathbf{Z}^{n+1}\right) \leq \varepsilon .$$

Thus, for α' close to α, the sum (37) is up to an error term of order $\varepsilon \cdot \mathrm{const}$ equal to

$$\sum_{G' \in \mathcal{G}^1_{\alpha',\beta}} \int_{E^\circ_{0,\beta}} (u^+_{G'} - u^-_{G'})\beta F_p(x, u^+_{G'}, p_{G'})\,dx .$$

Therefore, via formula (37), the term $\sum B^1_{G'}(E^+_{\alpha',\beta} \setminus V_{G'})$ in (36) may be replaced by this last sum. Because of the additional error term 2ε·const of (37), the total error does not exceed 3ε·const.

Finally, the expression

$$\int_{\mathcal{F}_{\alpha',\beta}(E^\circ_{0,\beta})} \beta F_p(\overline{x}, \Psi_{\alpha',\beta})\,d\overline{x} + \sum_{G' \in \mathcal{G}^1_{\alpha',\beta}} \int_{E^\circ_{0,\beta}} (u^+_{G'} - u^-_{G'})\beta F_p(x, u^+_{G'}, p_{G'})\,dx$$

has to converge for $\alpha' \to \alpha$ to

$$\int_{\mathcal{F}_{\alpha,\beta}(E^\circ_{0,\beta})} \beta F_p(\overline{x}, \Psi_{\alpha,\beta})\,d\overline{x} .$$

by elementary analysis using the regularity of $u^\pm_{G'}$ and F. This proves (36) and with it part B). □

Proof of Lemma 10 According to Lemma 6, for the sequence $\alpha' \to \alpha$ in consideration we have $\lim_{\alpha' \to \alpha} \mathcal{M}_{\alpha',0} = \mathcal{M}_{\alpha,0}$. For every $G \in \mathcal{G}_\alpha$ and to every α' of the sequence choose $G' = G'(G, \alpha') \in \mathcal{G}_{\alpha'}$ such that $\lim_{\alpha' \to \alpha} G' = G$. Set

$$\mathcal{G}^\circ_{\alpha'} \doteq \{\, G'(G, \alpha') \,:\, G \in \mathcal{G}_\alpha \,\} \subseteq \mathcal{G}_{\alpha'} .$$

Now replace in Corollary 3 the rotation vector α by α'. Using inequality (12) and the fact that $\mathcal{G}^\circ_{\alpha'} \subseteq \mathcal{G}_{\alpha'}$ we get

$$B(\alpha',\beta)+B(\alpha',-\beta) \geq \sum_{G'\in G^\circ_{\alpha'}/\Gamma_{0,\beta}} B^+_{G'}(E_{\alpha',\beta})+B^-_{G'}(E_{\alpha',\beta}) .$$

By standard arguments one shows that the right-hand side gives

$$\lim_{\alpha'\to\alpha} \sum_{G'\in G^\circ_{\alpha'}/\Gamma_{0,\beta}} B^+_{G'}(E_{\alpha',\beta})+B^-_{G'}(E_{\alpha',\beta}) = \sum_{G\in \mathcal{G}_\alpha/\Gamma_{0,\beta}} B^+_G(E_{\alpha,\beta})+B^-_G(E_{\alpha,\beta}) .$$

The arguments are similar to the one in the proof of Lemma 9, part B) 1. Consider first the convergence of $\mathcal{M}_{\alpha',\beta}$ to $\mathcal{M}_{\alpha,\beta}$ on a compact set of the form Q_τ with $\tau > 0$ large enough in the sense of (32).

Note that a gap $G' \in G^\circ_{\alpha'}/\Gamma_{0,\beta}$ again may approximate with its translates orthogonal to β more than one $G \in \mathcal{G}_\alpha/\Gamma_{0,\beta}$. Since $\mathcal{G}^\circ_{\alpha'}/\Gamma_{0,\beta}$ precisely consists of those gaps in $\mathcal{G}_{\alpha'}/\Gamma_{0,\beta}$ which approximate some gap $G \in \mathcal{G}_\alpha/\Gamma_{0,\beta}$, equality will hold in the limits $\alpha' \to \alpha$ and $\tau \to \infty$. We omit the details.

Notice that if $\mathcal{B}_0$ is not α'-admissible, the periodicity domain $E_{\alpha',\beta}$ may be defined due to the requirements on α' in exactly the same manner.

Finally, the right hand side of the last equality above is equal to $B(\alpha,\beta)+B(\alpha,-\beta)$ according to Corollary 3. □

References

1. Anderson, R.D., Klee, V.L.: Convex functions and upper semi-continuous collections. Duke Math. J. **19**, 349–35 (1952)
2. Aubry, S.: The concept of Anti-Integrability: Definitions, Theorems and Applications to the Standard Map. In: Gehee, R., Meyer, K.R. (eds) Twist Mappings and Their Applications, pp 7–54. Springer: Berlin Heidelberg New York, 1992
3. Aubry, S.: The Devil's staircase Transformation in incommensurate Lattices. Lecture Notes in Math. **925**, 221–245 (1982)
4. Bangert, V.: A uniqueness theorem for $\mathbf{Z}^n$-periodic variational problems. Comment. Math. Helv. **62**, 511–531 (1987)
5. Bangert, V.: On minimal laminations of the torus. Ann. Inst. Henri Poincaré (Analyse non linéaire) **2**, 95–138 (1989)
6. Bangert, V.: The existence of gaps in minimal foliations. Aequationes Mathematicae **34**, 153–166 (1987)
7. Bangert, V.: Geodesic Rays, Busemann Functions and Monotone Twist Maps. Calc. Var. **2**, 49–63 (1994)
8. Bangert, V.: Laminations of 3-tori by least area surfaces. In: P.H.Rabinowitz, E.Zehnder (eds) Analysis, et cetera, Academic Press: New York, 1990
9. Blank, M.L.: Metric properties of minimal solutions of discrete periodical variational problems. Nonlinearity **2**, 1–22 (1989)
10. Burago, D.Yu.: Periodic Metrics. Advances in Soviet Mathemathics **9**, 205–210 (1992)
11. Fathi, A.: Expansiveness, Hyperbolicity and Hausdorff Dimension. Comm. Math. Phys. **126**, 249–262 (1989)
12. Herring, C.: Some Theorems on Free Energies of Crystal Surfaces. Physical Review **82**, 87–93 (1951)
13. Le Calvez, P.: Les ensembles d'Aubry-Mather d'un difféomorphisme conservatif de l'anneau déviant la verticale sont en général hyperboliques. em C. R. Acad. Sci. Paris, t.306, Série I, 51–54 (1988)
14. Mather, J.N.: Differentiability of the Minimal Average Action as a Function of the Rotation Number. Bol. Soc. Bras. Mat. **21**, 59–70 (1990)

15. Moser, J.: Minimal solutions of variational problems on a torus. Ann. Inst. Henri Poincaré (Analyse non linéaire) **3**, 229–272 (1986)
16. Moser, J.: Minimal Foliation on a Torus. In: Giaquinta, M. (ed.) Topics in Calculus of Variations (Montecatini Terme 1987), pp 62–99. Springer Lecture Notes in Math. **1365** (1989)
17. Moser, J.: A stability theorem for minimal foliations on a torus. Ergod. Th. Dynam. Syst. **8***, 251–281 (1988)
18. Moser, J.: Monotone twist maps and the calculus of variations. Ergod. Th. Dynam. Syst. **6**, 401–413 (1986)
19. Rockafellar, R.T.: Convex Analysis. Princeton University Press: Princeton, 1970
20. Senn, W.: Strikte Konvexität für Variationsprobleme auf dem n-dimensionalen Torus. Manus. Math. **71**, 45–65 (1991)
21. Senn, W.: Über Mosers regularisiertes Variationsproblem für minimale Blätterungen des n-dimensionalen Torus. J. Appl. Math. (ZAMP) **42**, 527–546 (1991)
22. Senn, W.: Differentiability properties of the minimal average action. Master thesis (Adviser: V. Bangert), Universität Bern, Sidlerstr. 5, CH-3012 Bern, 1993
23. Vallet, F.: Thermodynamique unidimensionelle et structures bidimensionelles de quelques modèles pour des systèmes incommensurables. Thèse de Doctorat (Adviser: S. Aubry), Université Paris 6, 1986